

Dynamical billiards

Vladimir Georgiev, Irena Georgieva, Veneta Nedyalkova

1 Short introduction

Dynamical billiard is a dynamical system corresponding to the inertial motion of a point mass within a region that has a piecewise smooth boundary with elastic reflections. Billiards appear as natural models in many problems of optics, acoustics and classical mechanics. The most prominent model of statistical mechanics, the Boltzmann gas of elastically colliding hard balls in a box can be easily reduced to a billiard.

The billiard dynamical system is generated by the free motion of a mass-point (called a billiard ball) subject to the elastic reflection in the boundary. This means that the point moves along a geodesic line with a constant (say, unit) speed until it hits the boundary. At a smooth boundary point the billiard ball reflects so that the tangential component of its velocity remains the same, while the normal component changes its sign. In dimension two this collision is described by a well known law of geometrical optics: the angle of incidence equals the angle of reflection. Thus the theory of billiards and the theory of geometrical optics have many features in common.

One of the interesting billiard tables having elliptic shape is given on Figure 6



Figure 1: Elliptic table

The simplest billiard table is a circular one. Let $\kappa = \kappa(O, r)$ is a circle with center O and radius $r > 0$ (see Figure 2). If S_0 is a point on the circle κ in Figure 2 and it is starting point of the ray having S_1 as next intersection point, then at this point S_1 we have a reflection with the angle of incidence α_1 equals the angle of reflection β_2 . After the reflection the ray continues and we have the next intersection point S_2 of the ray, where again a reflection occurs and we have

$$\alpha_2 = \beta_2.$$

Each trajectory is defined by the initial points S_0 and the points

$$S_1, S_2, \dots$$

of the reflection at the boundary (in this case the circle κ).

The first observation is that at the points of reflections S_1, S_2, \dots we have

$$\alpha_1 = \beta_1 = \alpha_2 = \beta_2 = \dots,$$

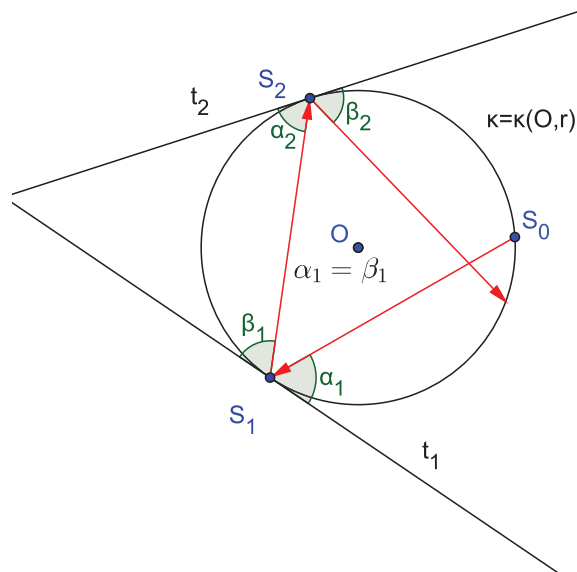


Figure 2: Circular billiard table

i.e. each trajectory makes a constant angle with the boundary (in this case is the circle κ).

On the other hand, the same trajectory remains tangent to a concentric circle. To clarify this fact we use the corresponding GeoGebra application.

In this application a new possibility of combining Java and Geogebra is used (see the Appendix).

One can see that each trajectory remains tangent to a concentric circle.

If S_0 is a point on the ellipse in Figure 5 and it is starting point of the ray having S_1 as next intersection point, then at this point S_1 we have a reflection with the angle of incidence α equals the angle of reflection β . After the reflection the ray continues and we have the next intersection point S_2 of the ray, where again a reflection occurs.

Before we proceed any further we introduce a new concept.

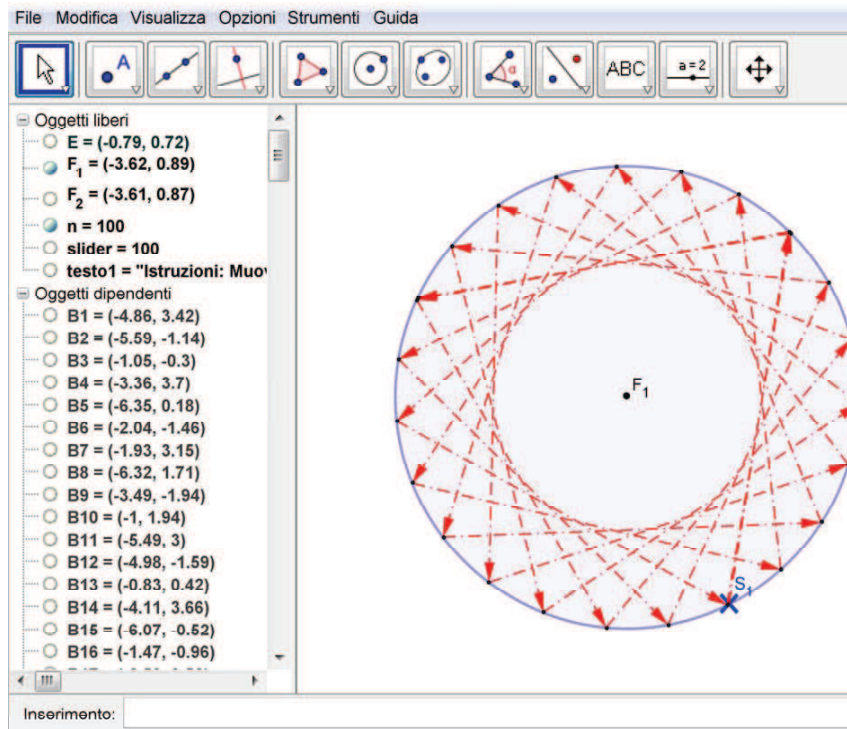
Definition 1. *A caustic of a plane billiard is a curve such that if a trajectory is tangent to it, then it again becomes tangent to it after every reflection.*

Thus the billiard in a circle has a family of caustics, consisting of concentric circles.

The next case to consider is that of conics. Recall that an ellipse consists of points whose sum of distances to two given points is fixed; these two points are called the foci of an ellipse. An ellipse can be constructed using a string, whose ends are fixed at the foci (the method carpenters and gardeners actually use). A hyperbola is defined similarly with the sum of distances replaced by the absolute value of their difference; and a parabola is the set of points at equal distances from a given point (the focus) and a given line (the directrix). Ellipses, hyperbolas and parabolas all have second order equations in Cartesian coordinates.

2 Some simple properties

The first result is the following optical property of ellipses.



Created with [GeoGebra](https://www.geogebra.org/m)

Figure 3: Geogebra application for Circle

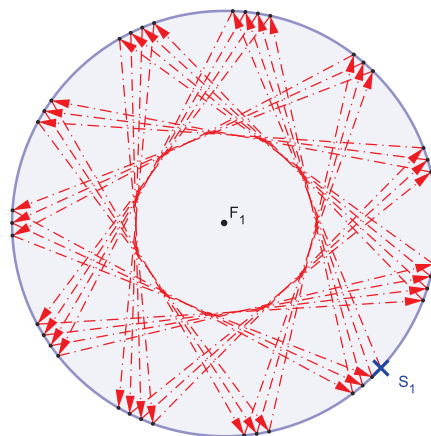


Figure 4: Geogebra application for Circle

Lemma 1. *A ray of light, emanating from one focus, comes to another focus after a reflection in the ellipse. Said otherwise, the segments, that join a point of an ellipse with its foci, make equal angles with the ellipse.*

Proof. Consider an extremal problem: given a line ℓ and two points F_1 and F_2 on one side of it, find a point X on ℓ such that the distance $|F_1X| + |XF_2|$ is minimal. Solution: reflect F_1 in

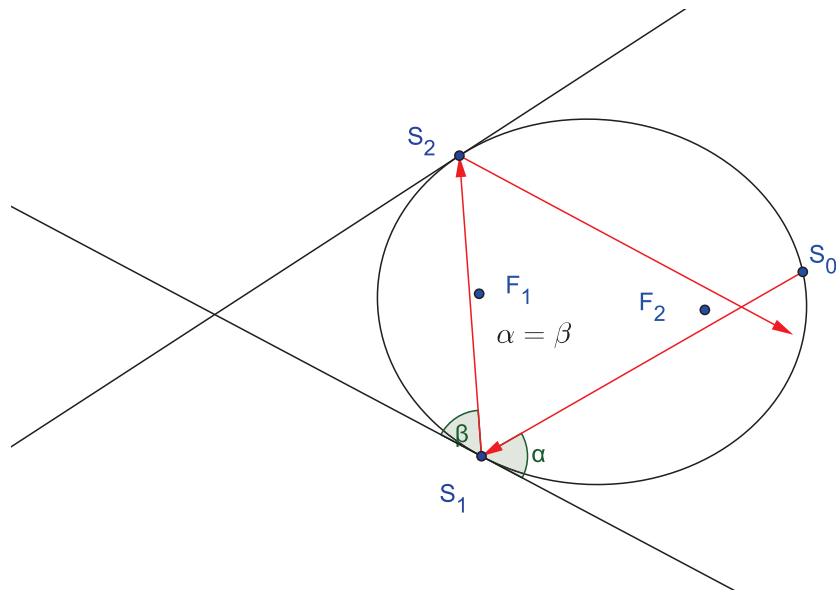


Figure 5: Billiard table

the line and join with F_2 by a straight segment.

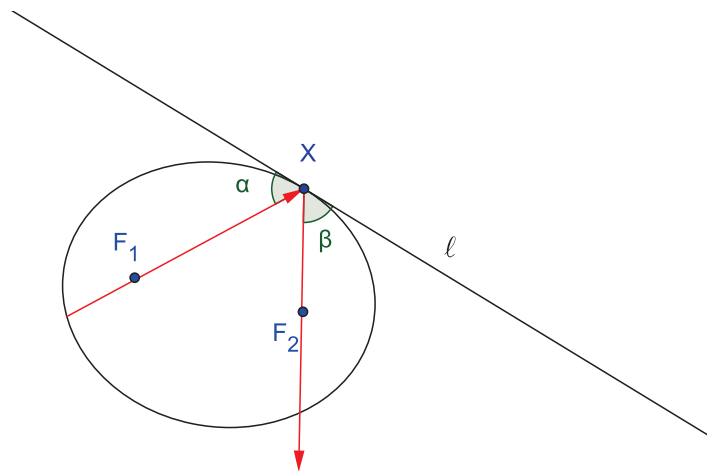


Figure 6: Basic property of ellipse

The point of intersection with ℓ is X . It follows that the angles made by F_1X and F_2X with ℓ are equal. On the other hand, X can be obtained as follows. Consider the family of ellipses with the fixed foci F_1 and F_2 . Then X is the point where an ellipse from this family touches ℓ for the first time. Hence X is the point of tangency of an ellipse with the foci F_1 and F_2 and

the line ℓ .

□

Likewise one proves the optical properties of a hyperbola and a parabola. These properties are extensively used in construction of various optical instruments.

Exercise 1. *If one puts a source of light in the focus of a parabolic mirror, then the reflected rays form a parallel beam (the property used in headlights' design).*

Ellipses and hyperbolas with the same foci are called confocal. In the appropriate Cartesian coordinates $(x; y)$ they are given by the equation:

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1, \quad (1)$$

with $0 < a < b$. Here λ is the variable parameter; for $-b^2 < \lambda < -a^2$ the curve is a hyperbola, and for $-a^2 < \lambda$ it is an ellipse.

Theorem 1. *An elliptic billiard table has a family of caustics, that consists of the confocal ellipses and hyperbolas. More precisely, if a segment of a billiard trajectory does not intersect the segment, joining foci F_1 and F_2 , then all the segments of this trajectory do not intersect F_1F_2 and are all tangent to the same ellipse with foci F_1 and F_2 ; if a segment of a trajectory intersects F_1F_2 , then all the segments of this trajectory intersect F_1F_2 and are all tangent to the same hyperbola with foci F_1 and F_2 .*

Proof. Let A_0A_1 and A_1A_2 be consecutive segments of a trajectory. Assume that A_0A_1 does not intersect the segment F_1F_2 (the other case is dealt with similarly). It follows from the optical property that the angles $\angle A_0A_1F_1$ and $\angle A_2A_1F_2$ are equal.

Reflect F_1 in A_0A_1 to F'_1 , and F_2 in A_1A_2 to F'_2 , and set: $G = F'_1F_2 \cap A_0A_1$; $H = F'_2F_1 \cap A_1A_2$. Consider the ellipse with foci F_1 and F_2 , that is tangent to A_0A_1 . Since the angles $\angle F_2GA_1$ and $\angle F_1GA_0$ are equal, this ellipse touches A_0A_1 at the point G . Likewise an ellipse with foci F_1 and F_2 touches A_1A_2 at the point H . One wants to show that these two ellipses coincide, or, equivalently, that $F_1B + BF_2 = F_1C + CF_2$, which boils down to $F'_1F_2 = F_1F'_2$. To this end one observes that the triangles $\triangle F'_1A_1F_2$ and $\triangle F_1A_1F'_2$ are congruent:

$$F'_1A_1 = F_1A_1; \quad F_2A_1 = F'_2A_1$$

by symmetry, and the angles $\angle F'_1A_1F_2$ and $\angle F_1A_1F'_2$ are equal. Hence

$$F'_1F_2 = F_1F'_2,$$

and the result follows. □

The following results have been obtained by using GeoGebra application. We have hyperbolic caustic on Figure 8 and elliptic ones on Figure 9.

3 Simplest periodic orbits in ellipse - triangles

As a first simulation one can use the Geogebra application to find triangle periodic orbit inside billiard table defined by the equation

$$e : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (2)$$

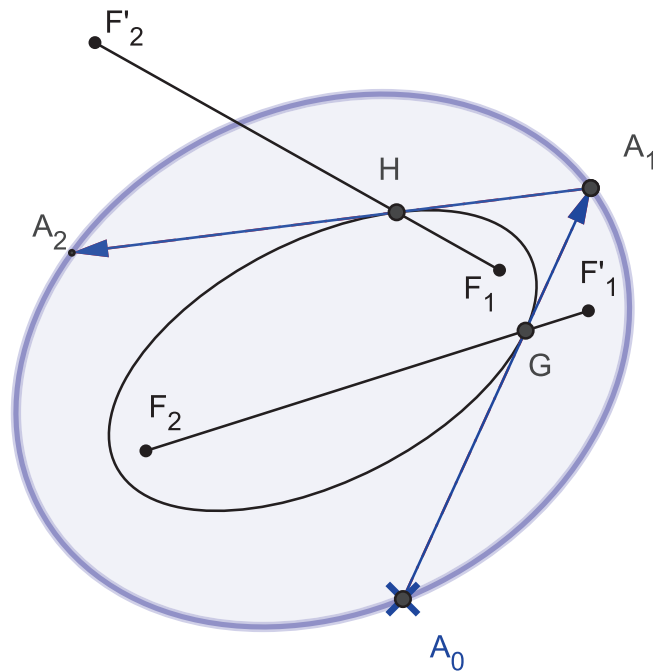


Figure 7: Caustics

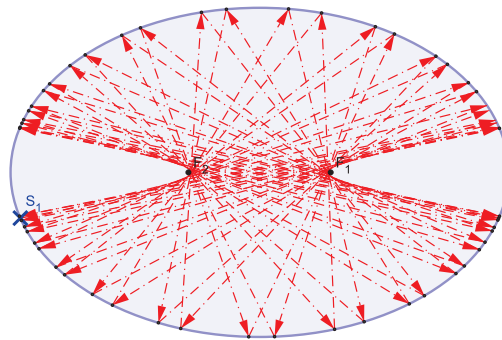


Figure 8: Hyperbolic Caustics

As a simplest initial point we choose the vertex of the triangle $\Delta A_0A_1A_2$ so that A_0 is on the y -axis and $A_0(0, b)$. Then it is natural to expect that if a periodic triangle exists, then (by symmetry with respect to y -axis) it has two equal sides ($A_0A_1 = A_0A_2$), see Figure 10.

However it is not clear how to find the point A_1 for example, since then A_2 is symmetric to A_1 with respect to the y -axis.

Recalling the assertion of Theorem 1, one can see that if $\Delta A_0A_1A_2$ is periodic, then its caustic

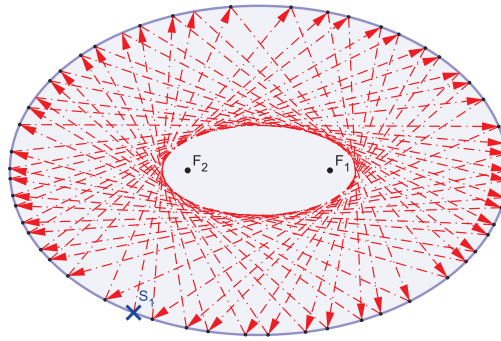


Figure 9: Elliptic Caustics

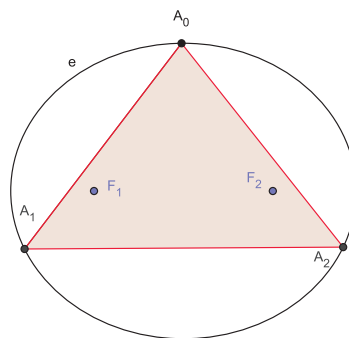


Figure 10: The problem to find periodic triangle from A_0 .

is a confocal ellipse, say

$$e_1 : \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1. \quad (3)$$

The equation (2) shows now that the condition that e and e_1 are confocal, i.e. have the same foci F_1 and F_2 can be expressed by

$$a^2 - b^2 = a_1^2 - b_1^2. \quad (4)$$

We shall suppose that e_1 is inside e so we have $a > b > 0, a_1 > b_1 > 0, a > a_1, b > b_1$. In this way we can reformulate our question:

- Given point $A_0(0, b)$ find an ellipse e_1 of type (3) inside the ellipse e so that the two tangents from A_0 to e_1 generate $\Delta A_0A_1A_2$ inscribed in e and circumscribed around e_1 (see Figure 11).

Even now the solution is not obvious and one should be very careful to avoid heavy and useless calculations that have no clear idea as a basis. So what to do? One can look in Internet and see that most of the documents found their are not very useful for High School teachers and students. Let us underline our main purpose: to use some concrete "TOOLS" as: algebraic manipulations, use of trigonometric functions, Geogebra applications and to try to pose and find solution to some interesting problems connected with the billiards on ellipse table.

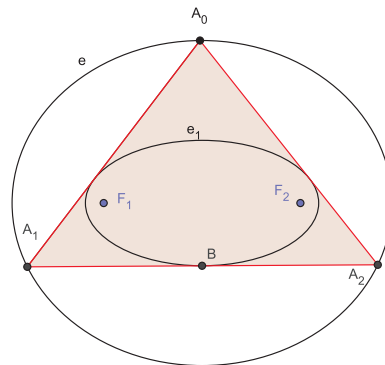


Figure 11: Caustic e_1 of the periodic triangle

So we try to generate several "simpler" questions and then we shall try to connect them and to clarify our strategy to approach the main problem of this section: to construct at least one periodic triangle explicitly in a given ellipse (2).

One possible list of questions is the following:

- Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and a line $y = kx + b$ through the point $A_0(0, b)$, find a necessary and sufficient condition (on k, b, a_1, b_1) so that the line is tangent to e_1 ;
- Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and a line $y - y_0 = k(x - x_0)$ through any point $A_0(x_0, y_0)$, find a necessary and sufficient condition (on k, x_0, y_0, a, b) so that the line is tangent to e_1 ;
- Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(0, b)$ find the tangent lines from A_0 to e_1 and find also the points A_1, A_2 of the intersection of these tangent lines with the ellipse $e : x^2/a^2 + y^2/b^2 = 1$ (we need formula expressing the coordinates of A_1, A_2 in terms of a, b, a_1, b_1);
- Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and the points A_1, A_2 described in the previous step find a necessary and sufficient condition (on a, b, a_1, b_1) so that the line A_1A_2 is tangent to e_1 ;
- Using the relation of the previous step as well the fact that e, e_1 have the same foci, i.e. $a^2 - b^2 = a_1^2 - b_1^2$ express a_1, b_1 in terms of a, b .

Each of these steps is not very difficult and we give the main points in the solution leaving some of the repeated details to the reader. The results are ordered in few Lemmas presented in the Appendix.

4 Possible further steps to find more periodic triangles

One can use a trivial symmetry and to see the taking $A'_0(0, -b)$ symmetric to A_0 with respect to the x - axis we have another periodic triangle symmetric to the original one $\Delta A_0A_1A_2$ from Lemma 6. The next step is to chose different starting point for the periodic trajectory and repeat the previous program.

- Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(a, 0)$ denote by t_1, t_2 the tangent lines from A_0 to e_1 and by A_1, A_2 the points of the intersection of these tangent lines with the ellipse $e : x^2/a^2 + y^2/b^2 = 1$, such that $A_1(x_1, y_1), x_1 < 0, A_2(x_2, y_2), x_2 > 0$. Try to find the coordinates of A_1, A_2 ;
- Given an ellipse $e : x^2/a^2 + y^2/b^2 = 1$ and the point $A_0(a, 0)$ try to find an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and periodic triangle $\Delta A_0A_1A_2$ so that e_1 is the caustic of the periodic triangle. Try to express a_1, b_1 in terms of a, b .

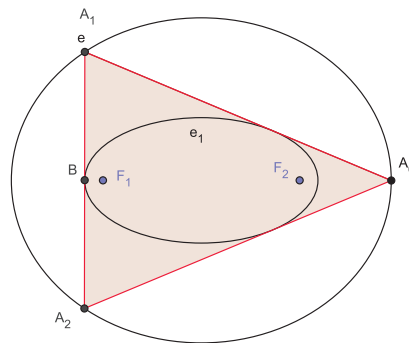


Figure 12: Another initial point A_0

One can see that the expressions for a_1, b_1 are the same as the expressions from Lemma 6.

Another extremely interesting Geogebra application is the activation of animation button on the point A_0 .

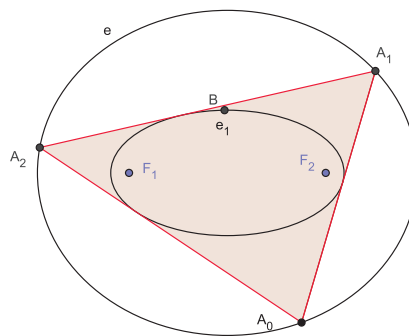


Figure 13: Surprise when A_0 is moving on e : tangent lines remain tangent to the caustic e_1 .

This simulation is very important since leads us to new open questions. We can make the following

Conjecture 1. *If A_0 is ANY point on the ellipse e , the small ellipse e_1 is defined according to Lemma 6 and the two tangent lines to e_1 from A_0 intersect the ellipse e in points A_1A_2 , then A_1A_2 is also tangent to e_1 .*

This phenomena is closely connected with the Poncelet Porisms (see [3], [6]). A solution of this problem can be found in [5].

Another application that is crucial for the further study of the billiards on the ellipse is the activation of some measurement instruments of Geogebra and to evaluate how vary the following quantities, when A_0 moves on the "orbit" of e :

- perimeter of the periodic triangle;
- area of the periodic triangle;
- angles of the periodic triangle.

After making this experiment one can discover (unfortunately only numerically!) the next amazing property.

Exercise 2. *If A_0 is ANY point on the ellipse e , then there exists a unique periodic triangle $\Delta A_0A_1A_2$ having constant perimeter, i.e. the perimeter is independent of position of the point A_0 on the ellipse e !!!*

We are not prepared at this moment to solve this Exercise, but one can make the following steps to verify partially the conjecture.

- take $A_0(0, b)$ and compute the perimeter P_1 of the corresponding periodic triangle;
- take $A_0(a, 0)$ and compute the perimeter P_3 of the corresponding periodic triangle;
- compare P_1 and P_2 .

One can verify that (see [4])

$$P_1 = P_2 = \frac{4a^2b(a + a_1)\sqrt{a^2 - a_1^2}}{b^2a_1^2 + a^2(a^2 - a_1^2)}.$$

5 Appendix I: Three technical lemmas

Lemma 2. *Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ one can express the necessary and sufficient condition such that the line $y = kx + b$ through the point $A_0(0, b)$ is tangent to e_1 as follows*

$$k^2 = \frac{b_2 - b_1^2}{a_1^2}.$$

Proof. Substituting y by $kx + b$ in (3) gives

$$\frac{x^2}{a_1^2} + \frac{k^2x^2 + 2kbx + b^2}{b_1^2} = 1. \quad (5)$$

The equation has only one real root so we need

$$\frac{b^2k^2}{b_1^4} - \left(\frac{1}{a_1^2} + \frac{k^2}{b_1^2}\right) \left(\frac{b^2}{b_1^2} - 1\right) = 0$$

and this identity is equivalent to

$$k^2 = \frac{b_2 - b_1^2}{a_1^2}.$$

This completes the proof of the Lemma. \square

Lemma 3. *Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ one can express the necessary and sufficient condition such that the line $y - y_0 = k(x - x_0)$ through the point $A_0(x_0, y_0)$ is tangent to e_1 as follows*

$$(y_0 - kx_0)^2 = b_1^2 + k^2a_1^2.$$

Lemma 4. *Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(0, b)$ denote by t_1, t_2 the tangent lines from A_0 to e_1 and by A_1, A_2 the points of the intersection of these tangent lines with the ellipse $e : x^2/a^2 + y^2/b^2 = 1$, such that $A_1(x_1, y_1), x_1 < 0, A_2(x_2, y_2), x_2 > 0$. Then we have*

$$x_1 = -\frac{2a^2a_1b\sqrt{b^2 - b_1^2}}{a_1^2b^2 + a^2(b^2 - b_1^2)} = -x_2,$$

$$y_1 = -M = y_2, \quad M = \frac{a^2b(b^2 - b_1^2)}{a_1^2b^2 + a^2(b^2 - b_1^2)}.$$

Proof. Substituting y by $kx + b$ in (2), we find the equation

$$\frac{x^2}{a^2} + \frac{k^2x^2 + 2kbx + b^2}{b^2} = 1. \quad (6)$$

One of the roots is obviously 0 and the other root is

$$x_1 = -\frac{2ka^2b^2}{b(b^2 + k^2a^2)} = -\frac{2ka^2b}{(b^2 + k^2a^2)}.$$

We can use the expression for k

$$k = \pm \frac{\sqrt{b^2 - b_1^2}}{a_1}$$

and taking into account the fact that $x_1 < 0$ obtain

$$x_1 = -\frac{2a^2a_1b\sqrt{b^2 - b_1^2}}{a_1^2b^2 + a^2(b^2 - b_1^2)}.$$

Using the relation

$$y = kx + b$$

we obtain the expression for y_1

$$y_1 = -\frac{a^2b(b^2 - b_1^2)}{a_1^2b^2 + a^2(b^2 - b_1^2)}.$$

This completes the proof. \square

From the relation $M = b_1$ and the fact that e and e_1 are confocal can be used to prove the following.

Lemma 5. *Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(0, b)$ let $A_1(x_1, y_1), x_1 < 0, A_2(x_2, y_2), x_2 > 0$ are the points determined in Lemma 5. Then A_1A_2 is tangent to e_1 if and only if*

$$a_1 = \frac{a(\sqrt{a^4 - a^2b^2 + b^4} - b^2)}{a^2 - b^2},$$

$$b_1 = \frac{b(a^2 - \sqrt{a^4 - a^2b^2 + b^4})}{a^2 - b^2}.$$

To avoid accumulation of too much technical proofs of lemmas we quote [4] for the proof of this Lemma.

Practically, Lemma 5 answers the question to find the caustic of the periodic triangles in a given ellipse e . Since it seems to be difficult to find this simple answer in the literature or in Internet we made this effort to give the answer with completely elementary tools (algebraic manipulations only). One can compare this answer with the classical results due to Cayley in [1], [2], where elliptic integrals are used. The result is useful and can be used in some of algorithms in GeoGebra (or some other IT tools) applications connected with billiard tables. In this direction we can mention the following difficulty met when one tries to implement Java scripts in Geogebra and simulate a non periodic trajectory with $N \gg 1$ reflecting points. It turns out that the construction of bisectrix as Geogebra tool combined with Java script causes some limitation on $N, N \leq 100$.

We close the Appendix with the following variant of Lemma 5.

Lemma 6. *Given an ellipse $e : x^2/a^2 + y^2/b^2 = 1$ and the point $A_0(0, b)$ one can find a unique ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and a unique periodic triangle $\Delta A_0A_1A_2$ so that e_1 is the caustic of the periodic triangle. Moreover, we have*

$$a_1 = a \frac{\sqrt{s^4 - s^2 + 1} - s^2}{1 - s^2}, \quad b_1 = b \frac{1 - \sqrt{s^4 - s^2 + 1}}{1 - s^2},$$

where

$$s = \frac{b}{a} \in (0, 1).$$

6 Appendix 2: Java applet

GeoGebra applets use Java technology to combine an interactive geometry environment with the ability to directly enter equations and coordinates making it very useful in math education and math explorations. The basic idea of Geogebra's interface is to provide two presentations of each mathematical object in its algebra and graphics windows. If you change an object in one of these windows, its presentation in the other one will be immediately updated. Computer algebra systems (such as Mathematica, Maple, and so on, e.g.) and dynamic geometry software (such as Geometers Sketchpad, Cabri Geometry, and so on, e.g.) are powerful technological tools for teaching mathematics. Numerous research results suggest that these software packages can be used to encourage discovery, experimentation and visualization in traditional teaching of mathematics. However, researches suggest that, for the majority of teachers, the main problem is how to provide the technology necessary for the successful integration of technology into teaching.

GeoGebra has been rapidly gaining popularity among teachers and researchers around the world, because it is easy-to-use dynamic mathematics software that combines many aspects of different mathematical packages. In addition, because of its open-source nature, an extensive user community has developed around it.

GeoGebra has some possibilities for animation. Including more modules for animating in GeoGebra should become an important technical element for future versions. Future extensions of the software GeoGebra will surely include more symbolic features of computer algebra systems which will further increase possible complex applications in the mathematical analysis, and 3D extensions.

A problem that we met is connected with the use of Java applets to produce high number of reflecting points in the billiard problem.

We use the following script associated to the slider in the program that has the following form

```
function stopAll(){
ggbApplet.evalCommand("StartAnimation[false]");
ggbApplet.setAnimating("slider",false);
ggbApplet.setValue("slider",100);}
var i= new Number(ggbApplet.getValue("slider"));
var lim = new Number(ggbApplet.getValue("n"));
if(i==0){
ggbApplet.evalCommand("a_{0}=
Vettore[S_{"+i+"},S_{"+(i+1)+"}]");
ggbApplet.setLineStyle("a_{0}",4);
ggbApplet.setColor("a_{0}",255,0,0);}
else if(i>=lim){
stopAll();}
else if(ggbApplet.exists("S_{"+(i)+"}")){
if(ggbApplet.evalCommand("bis"+(i)+"=
Bisettrice[F_1,S_{"+(i)+"},F_2]")){
ggbApplet.setVisible("bis"+(i),false);
if(ggbApplet.evalCommand("B"+(i)+"=
Intersezione[c, bis"+(i)+",2]")){
ggbApplet.setVisible("B"+(i),false);
if(ggbApplet.evalCommand("alpha"+(i)+"=
Angolo[S_{"+(i-1)+"},S_{"+(i)+"},B"+(i)+"}]"))
{ggbApplet.setVisible("alpha"+(i),false);
if(ggbApplet.evalCommand("R"+(i)+"=
Ruota[S_{"+(i-1)+"},2alpha"+(i)+",S_{"+(i)+"}]"))
{ggbApplet.setVisible("R"+(i),false);
if(ggbApplet.evalCommand("aa"+(i)+"=
Semiretta[S_{"+(i)+"},R"+(i)+"}]")){
ggbApplet.setVisible("aa"+(i),false);
if(ggbApplet.evalCommand("S_{"+(i+1)+"}=
Intersezione[c,aa"+(i)+",2]")){
ggbApplet.setPointSize("S_{"+(i+1)+"}",1);
ggbApplet.setLabelVisible("S_{"+(i+1)+"}",false)
if(ggbApplet.evalCommand("a_{"+(i)+"}=
Vettore[S_{"+i+"},S_{"+(i+1)+"}]")){
ggbApplet.setLineStyle("a_{"+(i)+"}",4);
ggbApplet.setColor("a_{"+(i)+"}",255,0,0);
}}}}}}
else {stopAll();}}
```

References

- [1] A. Cayley, *Note on the porism of the in-and-circumscribed polygon*, Philosophical magazine **6** (1853), 99102.
- [2] A. Cayley, *Developments on the porism of the in-and-circumscribed polygon*, Philosophical magazine **7** (1854), 339345.
- [3] V. Dragovic, M. Radnovic, *Poncelet Porisms and Beyond*, Birkhäuser, Springer-Basel, 2011.
- [4] V.Georgiev, V.Nedyalkova, *Perimeter of harmonic triangles in ellipse*, article in this book.
- [5] V.Georgiev, V.Nedyalkova, *Poncelet's porism and periodic triangles in ellipse*, article in this book.
- [6] S.Tabachnikov, *Geometry and Billiards*, Students Mathematica Library, (2005).