Piecewise defined functions

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1 Introduction

In applied problems piecewise defined functions often occur naturally e.g. functions describing prices of postage. These functions are also often used to demonstrate limits and discontinuities. Using a dynamic software like GeoGebra it is possible to study these functions in more detail and to use sliders to examine how their definition can be altered to create continuous functions, differentiable functions etc.

2 Defining functions on intervals in GeoGebra

In GeoGebra there are two ways to define functions on intervals; using the command *Function* or by using the command *If*.

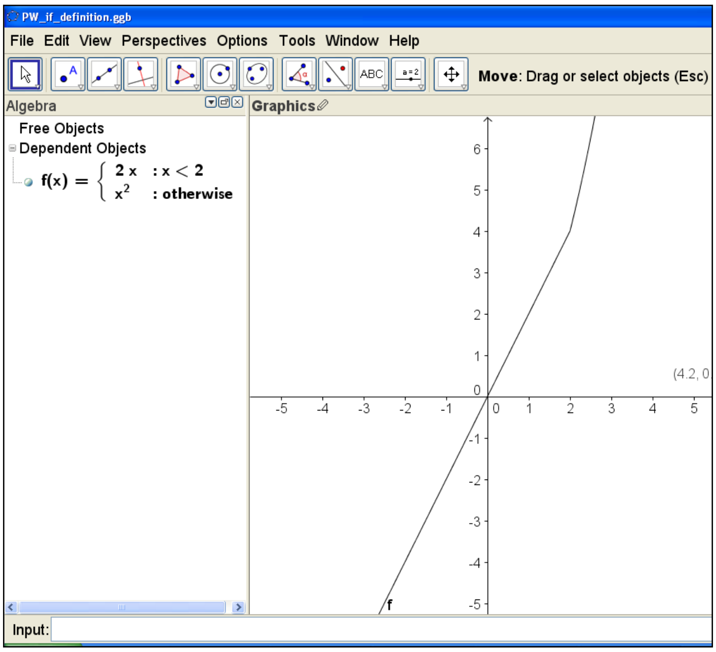


Fig.1 To define the function above we write *If[x < 2, 2x, x^2]* in the input field.

It is also possible to write *Function [2x,-∞,2]* and *Function[x^2,2,∞]* to get the same graph although this will result in two functions being defined. The *If* command has the advantage that the derivative of the function can be calculated directly.

If there are three different intervals a nested *If* command can be used e.g.

If[x < 2, x^2, If[x < 3, x + 2, 10 - x]] results in the figure below:

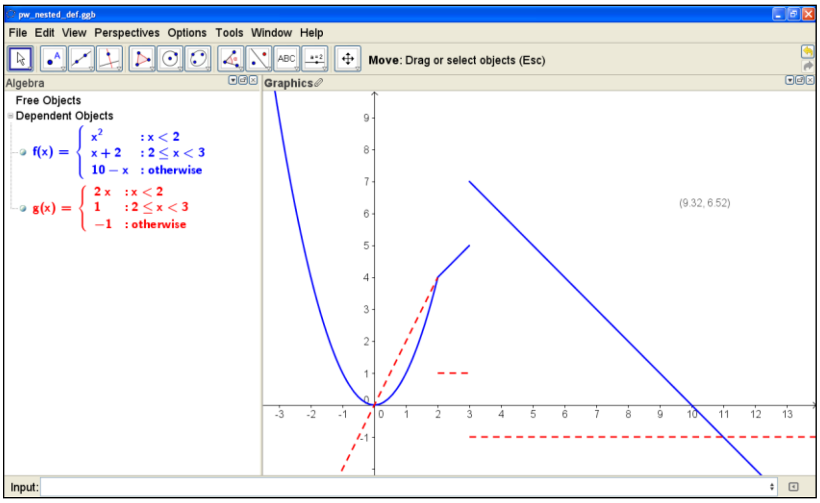
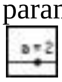


Fig. 2 A function *f(x)* defined on three intervals. Its derivative *g(x)* is shown in red. The derivative is undefined at $x = 2$ and at $x = 3$.

3 Sliders

Using GeoGebra it is very easy to investigate the effect of changing the value of one (or more) parameters occurring in the definition of a function. To define such a parameter select the slider tool  and click on the *Graphics view*. When that is done a small window opens:

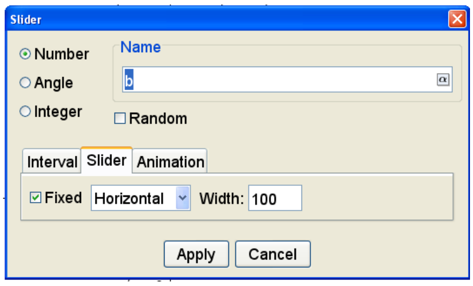


Fig. 3 This window is used for setting the interval of the parameter etc.

After a slider *b* has been defined it can be used in the definition of a function, e.g. $f(x) = x^2 + b$, and its value can be changed using the mouse.

4 Using sliders to define continuous functions

We consider the following problem: determine values of the parameters *a*, *b* and *c* such that the function

$$h(x) = \begin{cases} x - a & \text{if } x < -1 \\ x^2 + b & \text{if } -1 \leq x \leq 3 \\ c - x & \text{otherwise} \end{cases}$$

is continuous.

This we solve by defining three sliders *a*, *b*, *c* and using a nested *If* command to define *h(x)*. The sliders can easily be moved to make the function continuous.

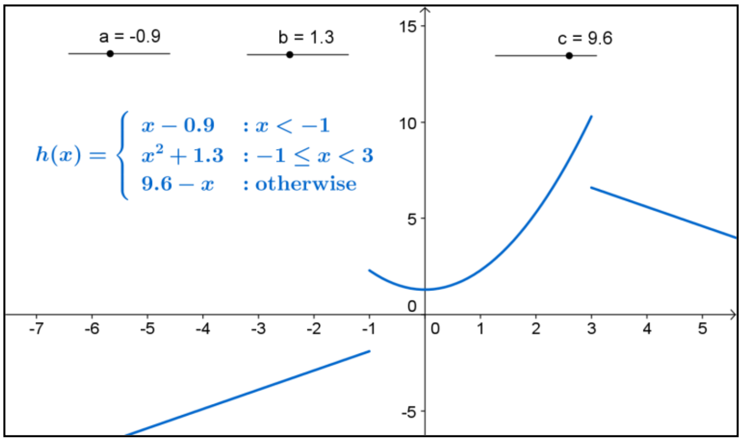


Fig. 4 We change the definition of the function by moving the sliders.

If we change the values of *a*, *b* and *c* the graphs move up and down so it is very easy to find values such that the graphs are connected.

Task: Define the function above in GeoGebra and find the appropriate values of the sliders. Do you get more than one solution? Solve the problem algebraically as well.

A more complicated situation arises if the parameters not only have to do with the location of the graph but also its shape.

Task: Define the function

$$g(x) = \begin{cases} x + 5 & \text{if } x < -1 \\ ax^2 + b & \text{if } -1 \leq x \leq 3 \\ 10 - x & \text{otherwise} \end{cases}$$

in GeoGebra and find values of *a* and *b* such that the function is continuous. Solve the problem also algebraically.

5 Differentiable functions

In problems like the ones given above it is very noticeable that at the connecting points we get a corner or a break in the function. This is because even though the function is continuous at the points it is not *differentiable*.

The derivative of a function $f(x)$ at $x = a$ is defined as the limit of the quotient of $\frac{f(a+h)-f(a)}{h}$ as h goes to 0 and the function is differentiable at $x = a$ if the limit exist. This limit is the slope of a tangent line to the graph of $f(x)$ at the point $(a, f(a))$. For the limit to exist it needs to be the same whether we approach the value a of x from the right or from the left. We can think of the limit as the slopes of secant lines going through the points $(a, f(a))$ and $(a+h, f(a+h))$ approaching the tangent line as h goes to 0. In the cases we have a break in the function these limit exist from the left and from the right but they are not equal.

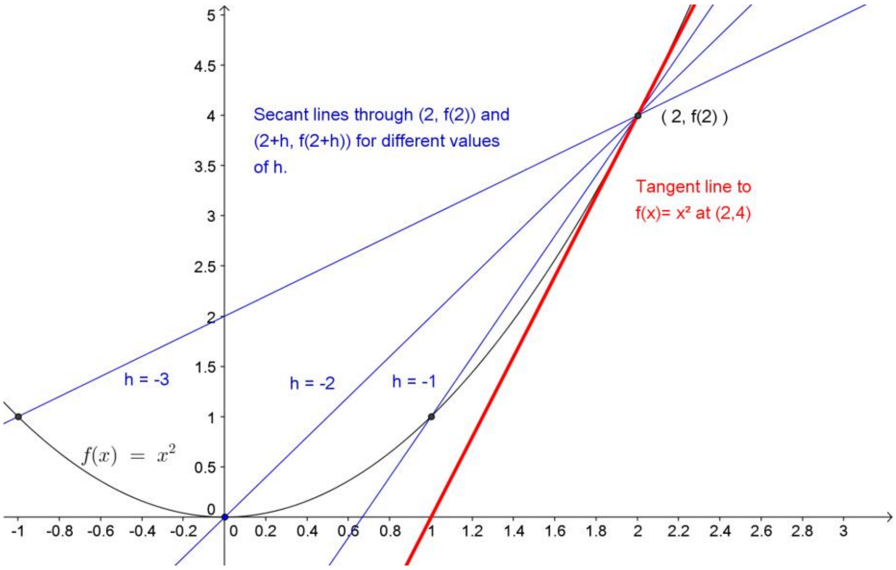


Fig. 5 Secant lines approaching the tangent line at a certain point

Task: Demonstrate the above by defining the tangent to *f(x)* at *x* = 2 (using the tangent tool), defining a slider *h* and a line through the points $(2+\ h, f(2\ +\ h\ ))$ and $(2, f(2))$ and then using the mouse to change the value of *h* and watch the secant approach the tangent.

Below we describe how we can join two different quadratic functions to get a piecewise defined function that is both continuous and differentiable at the meeting point.

Open a GeoGebra worksheet and define four sliders *b*, *c*, *d* and e . Define two quadratic functions $f(x) = -x^2 + bx + c$ and $g(x) = x^2 + dx + e$ and find values of the sliders such that $(1, f(1)) = (1, g(1))$. This ensures that the graphs of the two functions intersect at this point. Now use the tangent tool to get the tangents to both functions at the meeting point.

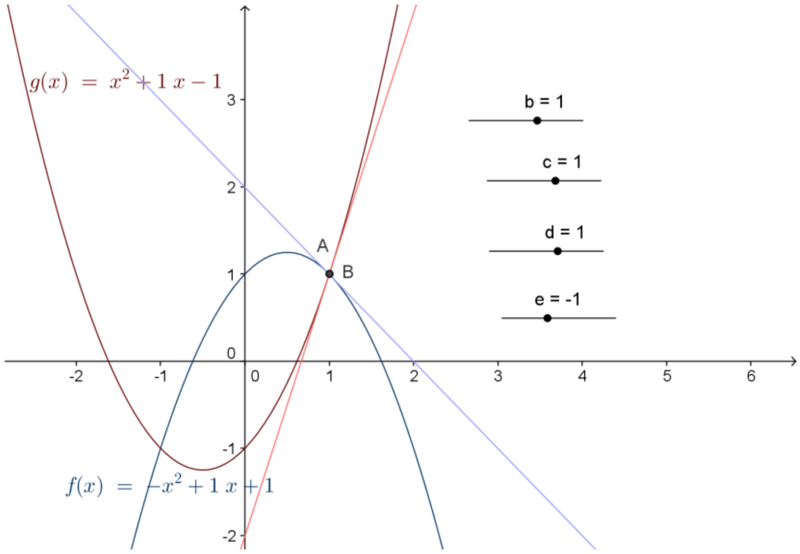


Fig.6 The graphs intersect at the $x = 1$ but the tangents are not the same.

If we now define a piecewise function $h(x)$ such that $h(x) = \begin{cases} f(x) & \text{if } x \le 1 \\ g(x) & \text{otherwise} \end{cases}$ we get the function below:

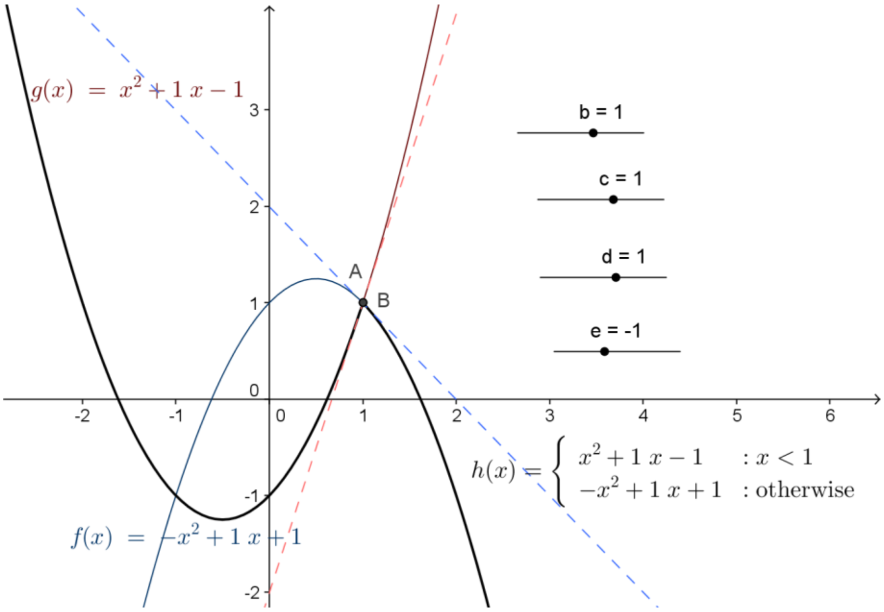


Fig. 7 The function $h(x)$ (black graph) is not differentiable at $x = 1$.

To get the function $h(x)$ to be differentiable we need to change the values of the sliders such that the tangents are the same.

Task: create the worksheet above and find values of the sliders such that the function *h(x)* is differentiable at every point.

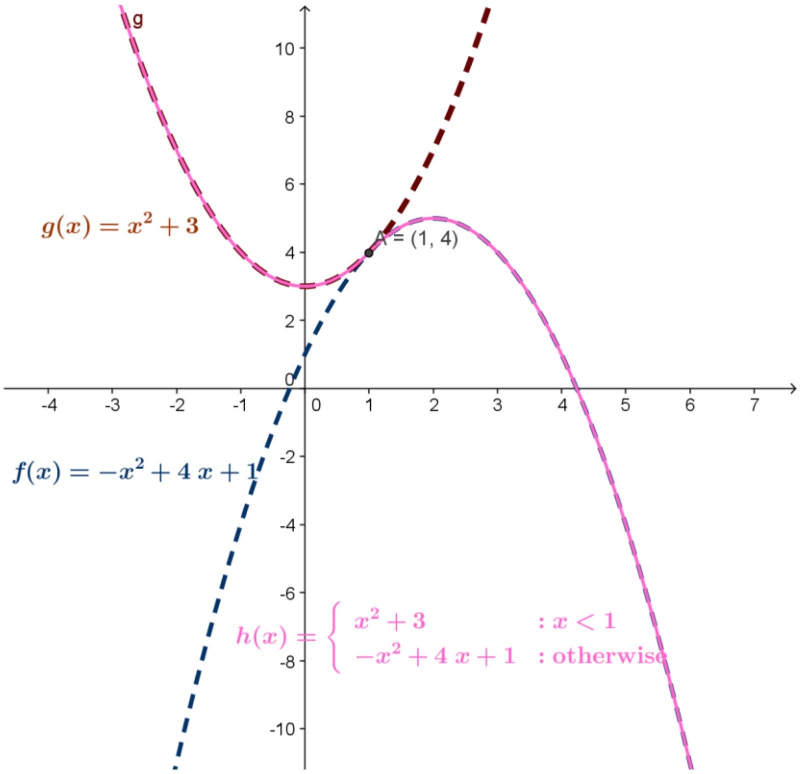


Fig. 8 Here we have one solution to the problem. The function $h(x)$ (pink graph) is differentiable everywhere.

7 Connecting second and third degree polynomials

We can make constructions similar to the ones above using other types of functions e.g. second and third degree polynomials as seen below.

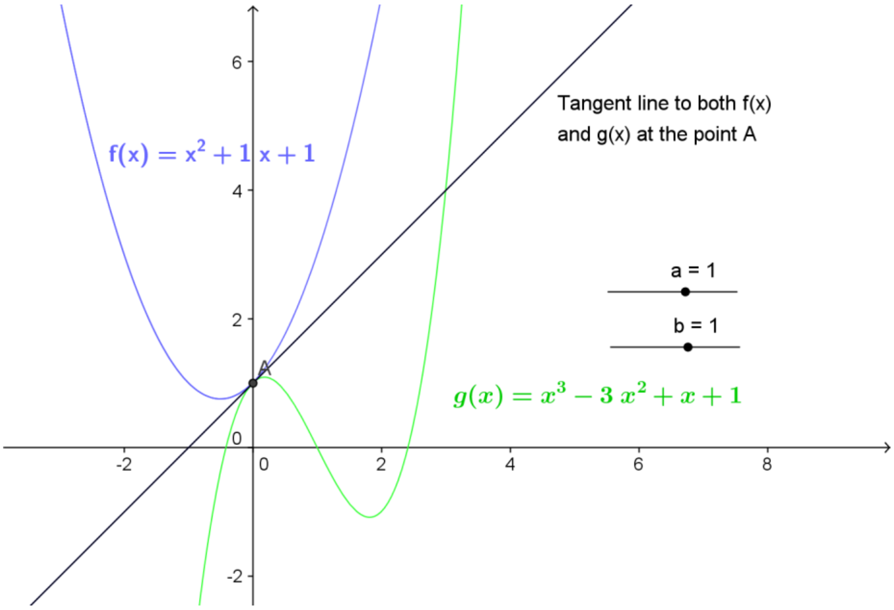


Fig. 9

The functions $f(x)$ and $g(x)$ have a common tangent at the point A so we can define a differentiable function using pieces of $f(x)$ and $g(x)$ on intervals separated by $x = 0$.

Task: make your own examples like this using e.g. two third degree polynomials.

6 Removing breaks

We can use a similar method to redefine a function on a small interval in order to remove a break in the graph of a function.

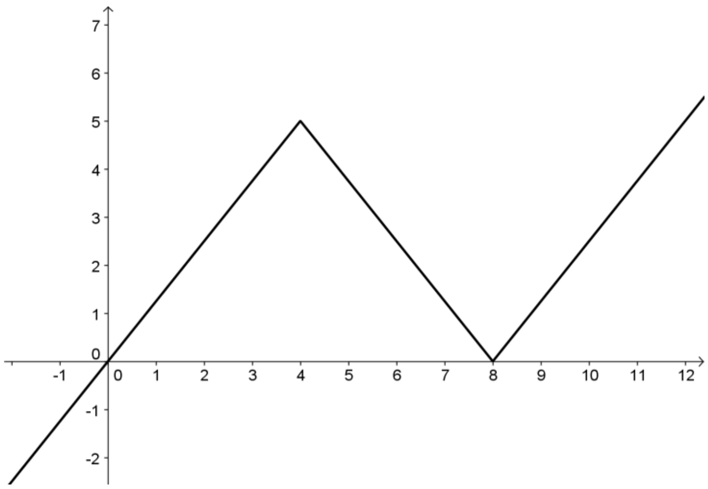


Fig. 10 A simple example of a function that has breaks at several points

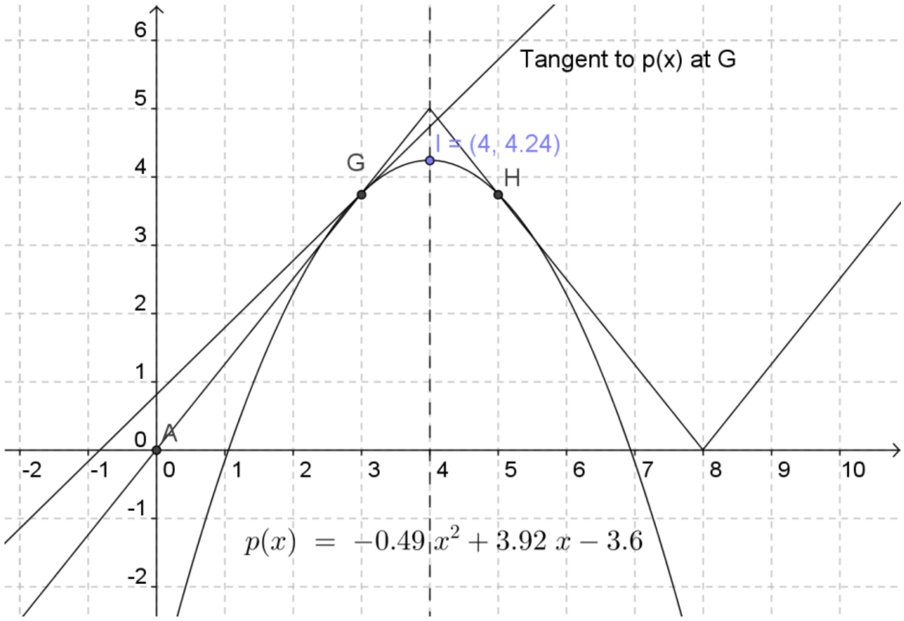


Fig. 11

We can define two points *G* and *H* on opposite sites of the break we want to remove and a point *I* on the line *x = 4*. We then use the command *FitPoly[{G,I,H},2]* to get a second degree polynomial that goes through these three points and the tangent tool to get a tangent to the graph of this polynomial at the point G. We then move the point *I* (it is fixed on the line $x = 4$) until this tangent coincides with the segment from (0,0) to (4,5). After removing help lines, changing colours etc we get the function below.

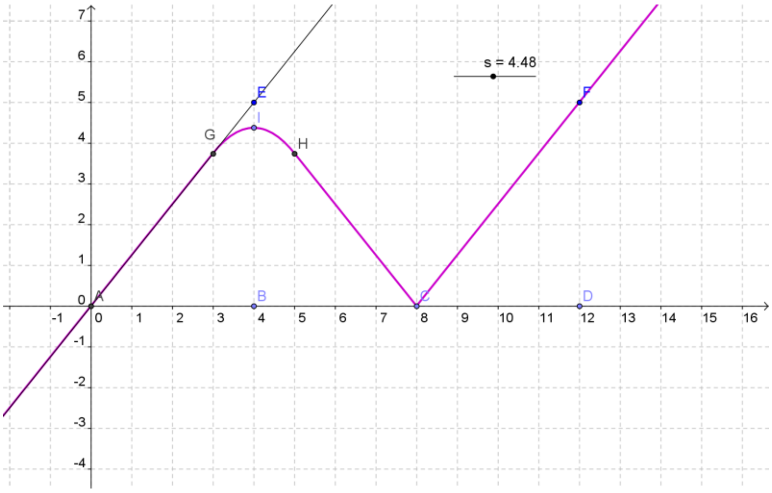


Fig. 12

Task: create the construction above

References

[1] GeoGebra, downloadable from http://www.geogebra.org.