Appendix II

The Story of a Project...

... or how GeoGebra can help in a difficult situation

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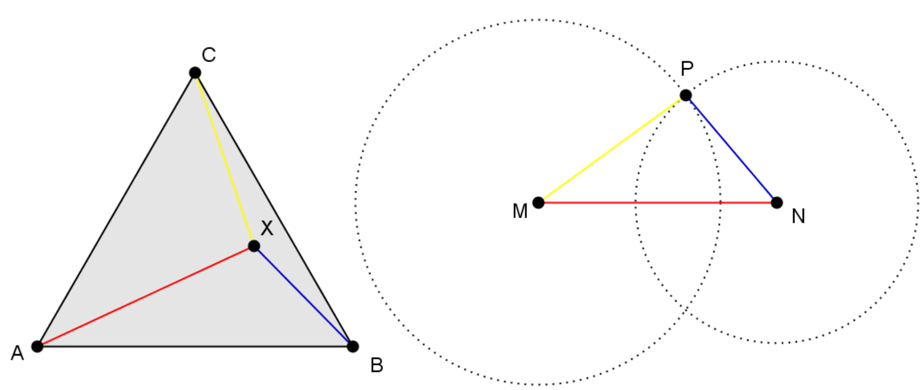
Everyone who has participated in a math project (regardless whether it was in the role of a teacher or in the role of a student), knows that a good project involves both intuition and imagination. Even knowledge is not that essential to start with, as long as one is willing to learn and grow – this way knowledge can be acquired in the process of working on the project.

But the same way an artist needs brushes, paint, and a canvas to create a painting, a project needs an appropriate environment to emerge.

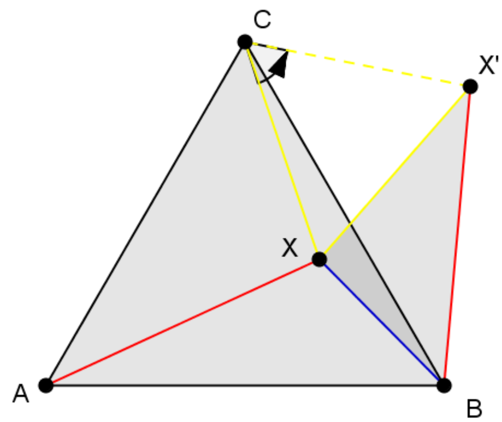
My name is Yanitsa, and instead of a brush, I used *GeoGebra* [1] to "paint" my project.

Last academic year I presented my "Pompeïu's Triangle" project at the Winter- and the Spring conferences of the *High School Institute of Mathematics and Informatics* (HSSI) [2]. Since there was surprisingly little literature on the subject, a significant part of my project consists entirely of my own research. The main geometric construction I worked with consists of an equilateral triangle ABC and a point X in its plane. According to Pompeïu's Theorem [3], the distances XA, XB, and XC from the point to the vertices of the triangle form the sides of a (possibly degenerate) triangle, i.e., they satisfy the known triangle inequality. This triangle is known as *Pompeïu's triangle*. A long time after I learned of this theorem, I tried to construct Pompeïu's triangle in various ways, with different positions of the point, so as to gain full understanding of the essence of the theorem. Of course, it turned out there are two ways for this geometric construction – an "obvious" one, and a "tricky" one.

Using the obvious approach, one simply takes the distances from the point X to the vertices of the triangle and constructs a new triangle given three sides. This is what the construction looks like:



Using the "tricky" approach though, one constructs Pompeïu's triangle as part of the initial configuration of the triangle ABC and point X, rather than separately, on the side, as we did above:

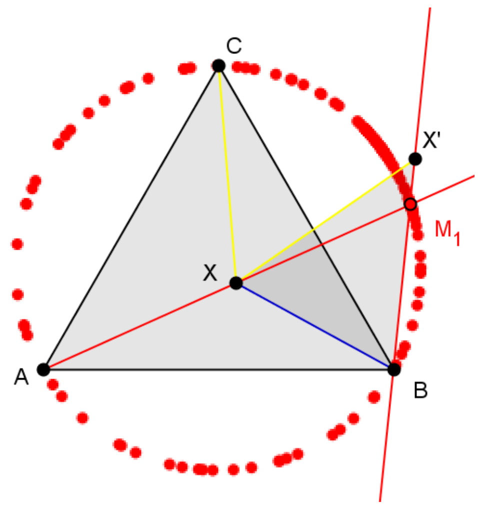


The point X' is the image of the point X under rotation of 60^{o} about one of the vertices of the ΔABC, and the sides of the ΔBXX' have lengths equal to the lengths of AX, BX and CX.

Even though this geometric construction contains only two new elements, I could easily build upon it in the search of previously unknown properties. What I added to it were a few points of intersection, after which a few triangles, then another triangle... I called this sequence of steps the "Proposition and the three corollaries". The formulation of the conjectures, which later became theorems, was entirely due to *GeoGebra*.

Conjecture 1

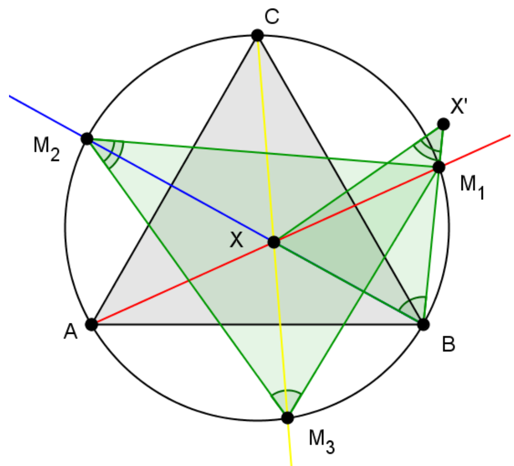
Let ΔXBX' be Pompeïu's triangle for ΔABC and the point X. Then, the intersection point of the ray AX and the line BX' lies on the circumcircle of ΔABC .



I could easily verify this, once I let *GeoGebra* trace out the intersection point M_1. Thus, the circumcircle of the ΔABC was formed.

Conjecture 2

If in an analogous way we add the points of intersection, M_2 and M_3, of the rays BX and CX with the circumcircle of ΔABC, then $\Delta M_1M_2M_3$ is similar to Pompeïu's triangle.

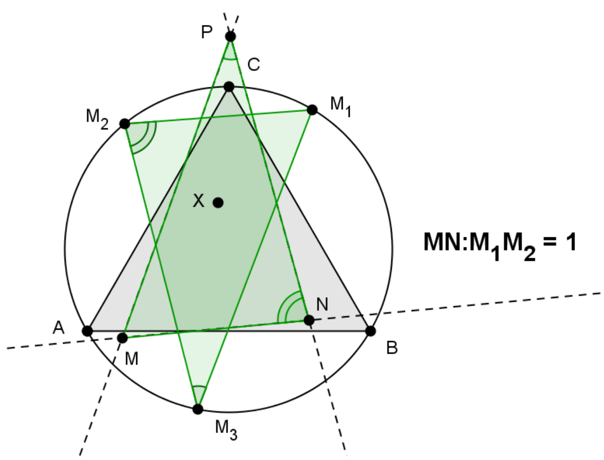


(The triangle $\Delta M_1M_2M_3$ constructed in this way is called a *circlecevian triangle*.**)**

To show the similarity of ΔABC and $\Delta M_1M_2M_3$, it suffices to show that two angles of $\Delta M_1M_2M_3$ are equal to two angles of Pompeïu's triangle. With this, *GeoGebra*'s job is done, since for the proof one cannot rely on any software

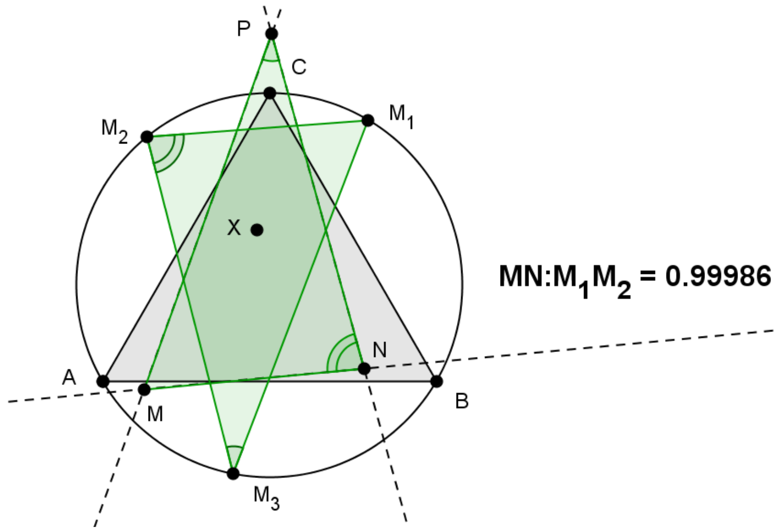
Conjecture 3

*The Simpson lines for the points M_1, M_2, and M_3 form a triangle which is congruent to the circlecevian triangle and therefore similar to Pompeïu's triangle***.**



In order to verify this conjecture, I constructed the three Simpson lines and considered the triangle formed by them.

To check if this triangle is congruent to the circlecevian triangle, I needed to make sure that the corresponding sides had equal lengths. These are nice results, right? Ratio 1:1. The picture changed, however, after increasing the floating point precision. This is in fact what was hiding behind this "1":



Without the dynamic sketch I created for this hypothesis, I would have lost days struggling to prove that these two triangles are congruent... when they in fact are "almost congruent". The ratio between their corresponding side lengths varies between 1.00003 and 1.15, which would have been impossible to tell had I drawn by hand.

After I realized the error in my conjecture, I corrected it by substituting "congruent" with "similar". This time everything worked out. Moreover, the proof turned out to be very beautiful!

Mathematical research is not easy. Sometimes certain "subtle" properties remain hidden in a sketch by hand. Or vice versa – some quite “obvious” facts turn out not to be true. No matter how talented a person is, the computer can always be of help. Whether it is *GeoGebra*, *SketchPad*, GEONExT or a similar software, in our fast changing computerized world, it would be a pity not to take advantage of the opportunities it can offer us. As I said in the beginning, any good mathematical project involves both intuition and imagination. Well, now I can add to this "a modicum of computer skills".

Reference

[1] *GeoGebra* http://www.geogebra.org/cms/

[2] *High School Institute of Mathematics and Informatics* http://www.math.bas.bg/hssi/

[3] Pompeïu's Theorem http://mathworld.wolfram.com/PompeiusTheorem.html