Dynamat and math projects

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23.7.2012

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Sofia 23 . 7. 2012

1. Initial point: Short presentation of Comenius project
2. Napoleon's problem
   - Some generalizations of Napoleon's theorem
3. Dynamical systems
   - Biliards

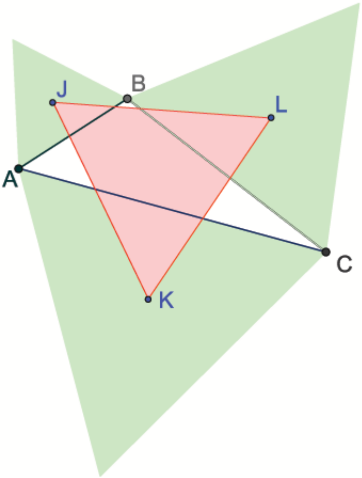
Title of the Comenius Project
Dynamical and creative Mathematics using ICT - Comenius project
Goal: to prepare concrete examples, hints and good practices in preparation of future and in service teachers to develop creativity of their pupils.

Partners:
Italy, University of Pisa
Austria, University of Vienna,
Denmark: VIA University College - Bachelor Programme in Teacher Education, Aarhus
Bulgaria: Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia
Slovakia: Constantine the Philosopher University Nitra, Nitra
Iceland: University of Iceland - School of Education, Reykjavik

The following statement (known as Napoleon's theorem) is closely connected with the Fermat theorem.

Theorem 1

(Napoleon's Theorem) On each side of a triangle an exterior equilateral triangle is constructed. Show that the centers of the three thus obtained equilateral triangles are vertices of an equilateral triangle.



Napoleon's theorem

Our interest to study Napoleon's theorem is partially motivated by the fact that it is not very popular among Italian Math professors. Although, it is difficult to explain this phenomenon, one can easily predict (and this was indeed verified in practice) that Math students and consequently future math teachers have no any idea about this beautiful math statement.

A well - known generalization of Napoleon's theorem is the following one

D. Wells, *You Are a Mathematician*, John Wiley & Sons, NY, 1981.

For an arbitrary $\triangle ABC$, three exterior points A_1, B_1, C_1 are constructed such that

$$\triangle ABC_1 \sim \triangle BCA_1 \sim CAB_1.$$

Then the centroids of these triangles are vertices of a triangle similar to them.

Actually it's not even necessary to consider the centroids.

For arbitrary $\triangle ABC$ take three external points A_1, B_1, C_1 such that

$$\sphericalangle AC_1B + \sphericalangle BA_1C + \sphericalangle CB_1A = 360^0.$$

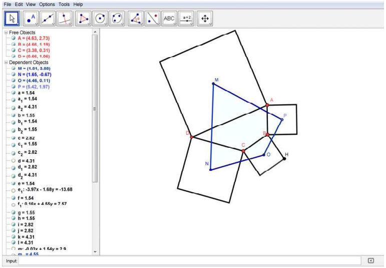
Then $\triangle A_1B_1C_1$ is similar to a triangle with angles

$$\sphericalangle C_1AB + \sphericalangle B_1AC, \sphericalangle C_1BA + \sphericalangle A_1BC, \sphericalangle A_1CB + \sphericalangle B_1CA.$$

One can see the references (
D. Wells, *You Are a Mathematician*, John Wiley & Sons, NY, 1981.
S. B. Gray, *Generalizing the Petr-Douglas-Neumann Theorem on N-Gons*, The American Mathematical Monthly, **110(3)** (2003), p. 210 – 227.
for the proof of this interesting fact.

Napoleon's Theorem quadrilaterals - GeoGebra Dynamic Worksheet Page 1 of 1

Napoleon's Theorem quadrilaterals



Vladimir, Created with GeoGebra

Geogebra application: Quadrilaterals and Napoleon’s theorem

Another generalization is due to Steve Gray

S. B. Gray, *Generalizing the Petr-Douglas-Neumann Theorem on N-Gons*, The American Mathematical Monthly, **110(3)** (2003), p. 210 – 227.

This time, the construction starts with an arbitrary $n-$gon and one considers the $n-$ gon formed by the centers of the regular $n-$gons constructed externally on its sides. As it was shown in

S. B. Gray, *Generalizing the Petr-Douglas-Neumann Theorem on N-Gons*, The American Mathematical Monthly, **110(3)** (2003), p. 210 – 227.

after repeating this construction $n-2$ times one obtains a regular $n-$gon.

Having in mind Gray's generalization of Napoleon theorem, one could ask if it is possible to obtain a regular $n-$ gon after less that $n-2$ steps. For example taking $n=4$ and using Geogebra one can see that in general it's impossible.

Hence, it is natural to ask for which $n-$gons one can obtain a regular $n-$ gon after the first step. We do not know the answer to this question for arbitrary n, but for the case $n = 4$ it is given in the next exercise.

Exercise 2

On the sides of a quadrilateral external squares are constructed. Prove that:
(a) The centers of the squares are vertices of a quadrilateral with perpendicular diagonals of equal length.
(b) The quadrilateral in (a) is a square if and only if the initial quadrilateral is a parallelogram.

We propose to the reader to prove the following generalization of Napoleon's theorem:

Exercise 3

On the sides of a non-equilateral triangle three regular $n-$gons are constructed externally to the triangle. Prove that

$$\text{their centers are vertices of an equilateral triangle iff } n = 3. \quad (1)$$

Billiards as dynamical systems

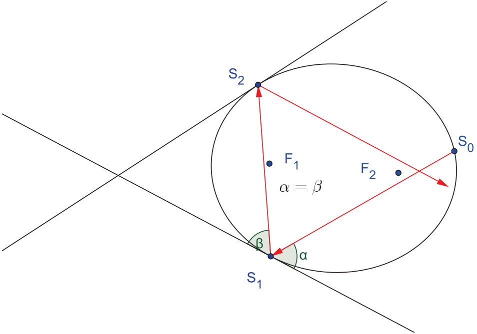


Figure: Billiard table

If S_0 is a point on the ellipse in Figure 1 and it is starting point of

Billiards as dynamical systems

A caustic of a plane billiard is a curve such that if a trajectory is tangent to it, then it again becomes tangent to it after every reflection.

The first result is the following optical property of ellipses.

Lemma

A ray of light, emanating from one focus, comes to another focus after a reflection in the ellipse. Said otherwise, the segments, that join a point of an ellipse with its foci, make equal angles with the ellipse.

Periodic triangles

Periodic triangle is any billiard trajectory such that $S_3 = S_0$, i.e. they are periodic trajectories.

Task 1 Periodic triangles starting at fixed point S_0 have the following optimal property: minimization of the triangles with vertex at S_0 inscribed in the given ellipse.

S.Tabachnikov, Geometry and Billiards, Students Mathematica Library, (2005).

If A_0 is ANY point on the ellipse e, then there exists a unique periodic triangle $\triangle A_0A_1A_2$ having constant perimeter, i.e. the perimeter is independent of position of the point A_0 on the ellipse e!!!

Reference mathoverflow:

Difficult^3 task If a convex curve satisfies the property that periodic triangles have constant perimeter, then the curve is ellipse.