

Around Napoleon's Theorem

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1 Napoleon's theorem: a little bit history

Napoleon was known to be an amateur mathematician. Among his friends was the Italian mathematician Lorenzo Mascheroni who introduced the limitation of using only a compass (no straight edge) in geometric constructions. One of Napoleon's problems in this direction is to find the center of a given circle using only a compass. The problem is not trivial, but since this is not the main purpose of our discussions, we shall mention only that it is interesting because problems of that kind are real life problems from the Renaissance period (1300-1600). The Renaissance inventions and discoveries of this period include for example:

- Mechanical Clock;
- Artillery, launching tubes introduced by an engineer William Congreve ;
- Printing Press, the machine was invented in 1440 by Johann Gutenberg of Germany;
- Compass, first used by a Chinese voyager Zheng He (1371-1435);
- Microscope, Hans Janssen developed the first compound microscope in 1509;
- Wallpaper, the first paper mill was set up in England in the year, 1496;
- Submarine, a design of submarine was created by Leonardo Da Vinci. However, Cornelius van Drebbel invented the submarine in the year 1624;
- The Match, invention made in 1680 by Robert Boyle;
- Eyeglasses, Salvino D'Amate an Italian inventor developed the earliest form of eyeglasses in 1284.

The importance of these inventions and the fact that they changed significantly the real life explain why some important math problems from that period have been connected with some new practical issues.

Another example of this period: P. Fermat (1601-1665) challenged Evangelista Torricelli (1608-1647), the inventor of barometer with the following question

- Find a point in the plane such that the sum of their distances from the vertices of a triangle is a minimum.

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Torricelli presented several solutions to this problem. In one of them he observed that the circumcircles of the equilateral triangles constructed externally on the sides of a given triangle meet in a point (called now Fermat's point).

Remarkable math statements have been attributed to Napoleon Bonaparte (1769-1821) although his relation to the theorems and their proofs is questioned in most of the sources available to our knowledge. Nevertheless, the mathematics flourished in post-revolutionary France and mathematicians were held in great esteem in the new Empire. Laplace was a Minister of the Interior under Napoleon.

The following statement (known as Napoleon's theorem) is closely connected with the Fermat problem (see [14]), presented above.

Theorem 1. (*Napoleon's Theorem*) *On each side of a triangle an exterior equilateral triangle is constructed. Show that the centers of the three thus obtained equilateral triangles are vertices of an equilateral triangle.*

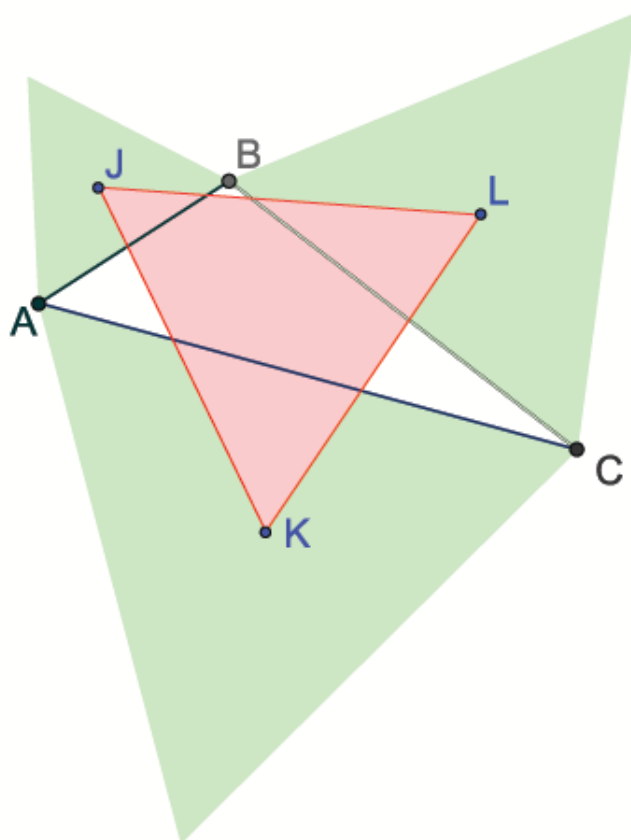


Figure 1: Napoleon's theorem

It's indeed quite surprising that the shape of the resulting triangle does not depend on the shape of the original one. However it depends on the shape of the constructed triangles: it's equilateral whenever the latter triangles are equilateral. The corresponding Geogebra file can be found on the following link [Napoleon problem](#).

This is our starting point. Our main purpose is to develop concrete didactic units (supported and interacting with Geogebra outputs) that can be used:

- in courses for preparation of future or in service teachers
- to implement some of materials in classroom practice.

Our interest to study Napoleon's theorem is partially motivated by the fact that it is not very popular among Italian Math professors. Although, it is difficult to explain this phenomenon, one can easily predict (and this was indeed verified in practice) that Math students and consequently future math teachers have no any idea about this beautiful math statement.

2 GeoGebra simulations of Napoleon's theorem

We use Geogebra since it has a friendly and easy way of preparing geometric applications. This allows us to move the points A, B, C freely as vertices of an arbitrary triangle.

The Geogebra application can be activated on the following link [Napoleon problem with triangles](#). It is prepared in collaboration with the student Sara Leal Venegas during the work of the Math lab "Problem Posing" in the spring of 2011 in Pisa University.

As a first example of questions that one may ask using the dynamic application is the following one:

- Is the ratio of the areas of $\triangle JKL$ and $\triangle ABC$ on Figure 1 constant?

This question can be easily answered by means of the Geogebra application. Moving the point A in such a way that in the limiting case A, B, C are collinear, one can keep the area of $\triangle JKL$, while the area of $\triangle ABC$ tends to 0. So the conclusion is that the answer is negative.

Another question related to above one is the following:

- Can we find the area of $\triangle JKL$ or equivalently can we find its side?

This example shows how one can "jump" from Geogebra application to an abstract problem whose solution needs pure math reasoning and computations without using IT tools.

Turning back to Napoleon's theorem we had experience during the work of the Problem Posing Lab with the following questions:

- What happens if one replaces the equilateral triangles by squares?
- Is there a generalization of Napoleon's theorem replacing the initial triangle by a quadrilateral ?

The answer to the first question was found very rapidly by using Geogebra application, (see Figure 2) that can be activated on the following link [Napoleon problem with squares](#).

One can see that if the point B approaches A , then $\triangle KLJ$ (see Figure 3) becomes very close to a right triangle, that is it is not equilateral.

Using the same Geogebra simulation one can see that the centroids of the triangles ABC and JKL coincide. Indeed, using Exercise 5 we see that the complex numbers corresponding to the vertices of these triangles satisfy the relations (see Figure 3)

$$\ell = \left(\frac{1 \pm i}{2}\right) b + \left(\frac{1 \mp i}{2}\right) c, \quad k = \left(\frac{1 \pm i}{2}\right) c + \left(\frac{1 \mp i}{2}\right) a, \quad (1)$$

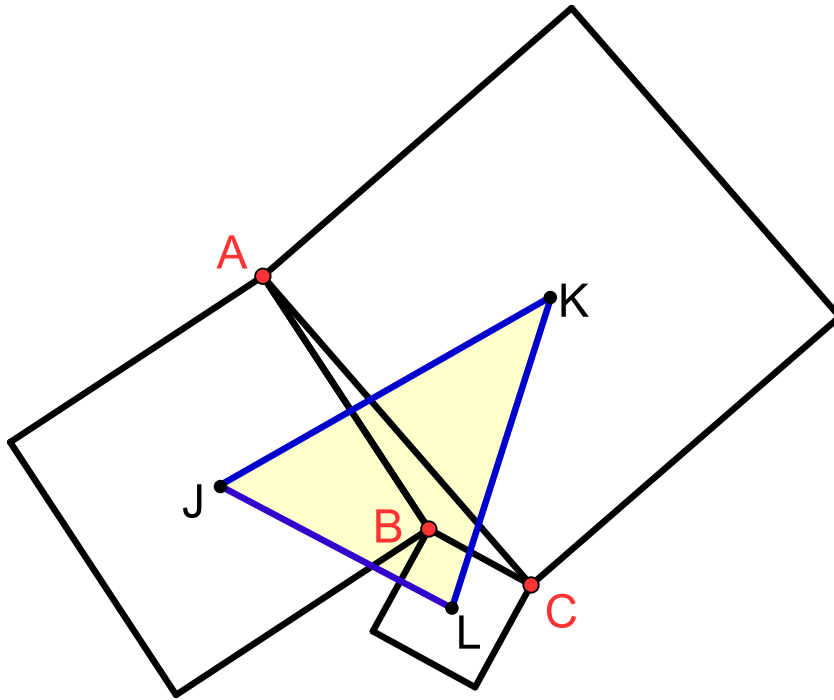


Figure 2: Geogebra replaces triangles with squares

$$j = \left(\frac{1 \pm i}{2}\right) a + \left(\frac{1 \mp i}{2}\right) b. \quad (2)$$

Now it is easy to check that

$$\ell + k + j = a + b + c,$$

so the centroids of $\triangle ABC$ and $\triangle LKJ$ coincide.

Since each of the triangles BCL , CAK and ABJ is isosceles right triangle, it is natural to ask the following question:

- Are there three distinct non - collinear points A, B, C such that $\triangle LKJ$ is an isosceles right triangle ?

The answer (suggested by the previous simulations with Geogebra) is the following

Lemma 1. *If A, B, C are three distinct non - collinear points, then $\triangle LKJ$ is not an isosceles right triangle.*

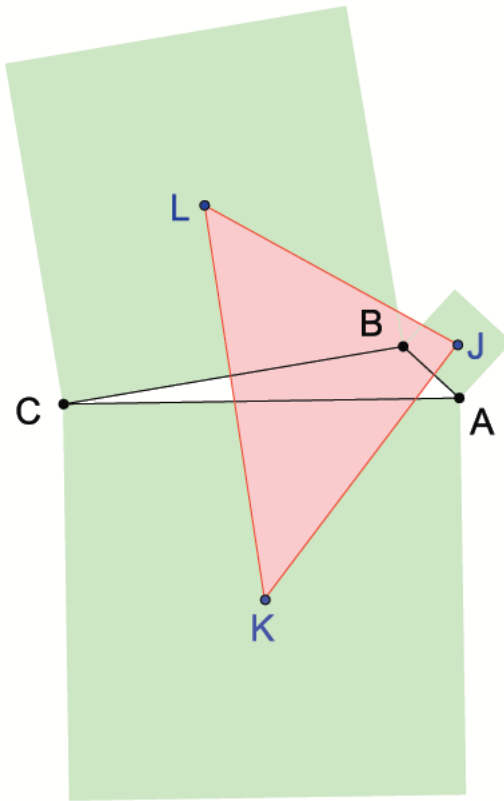


Figure 3: Geogebra counterexample for generalization of Napoleon's theorem

Proof. If we suppose that $\triangle JLK$ is an isosceles right triangle, then we have the relation

$$j = \frac{1 \mp i}{2} \ell + \frac{1 \pm i}{2} k.$$

Substituting (1) and (2) in this relation, we obtain

$$\frac{i}{2}(a - b) = 0$$

and this of course leads to a contradiction. □

Another question that may be posed is the following:

- Are there three distinct non - collinear points A, B, C , such that LKJ is an equilateral triangle ?

One can use Lemma 2 to solve the following:

Exercise 1. *If A, B, C are three distinct non - collinear points, then $\triangle LKJ$ is equilateral iff $\triangle ABC$ is equilateral .*

A more interesting question is to find all cases, when $\triangle LJK$ is a right triangle. We can state the following "conjecture" that has been tested with Geogebra simulations.

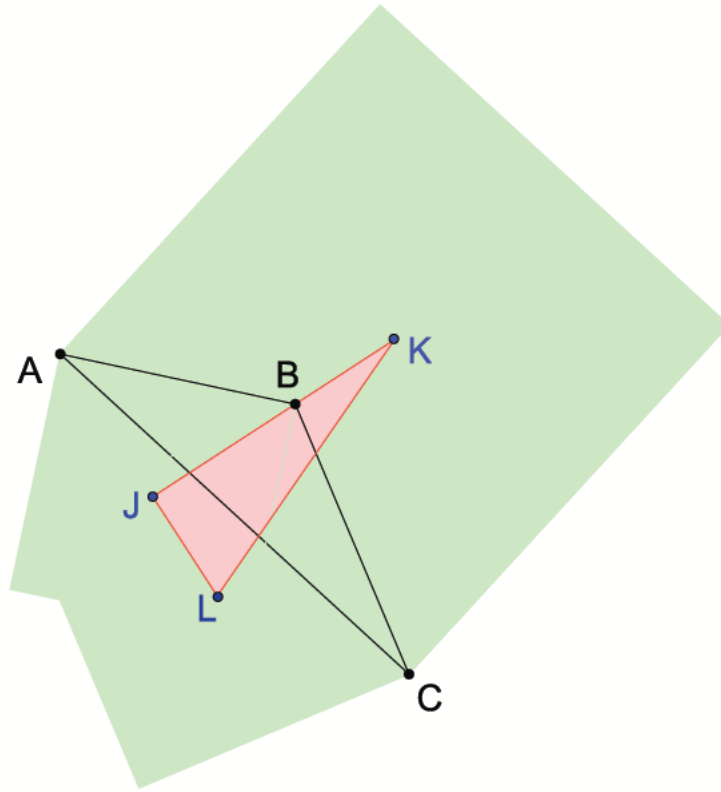


Figure 4: Right triangles with counterclockwise orientation of $\triangle LKJ$

Proposition 1. *If A, B, C are three distinct non - collinear points, then $\sphericalangle LJK = 90^\circ$ iff A and B lie on the lines LJ and KJ , respectively.*

Proof. Take an arbitrary $\triangle ABC$ with clockwise orientation, for example (see Figure 4). We suppose that the centroids L, K, J of the squares are such that $\triangle BCL$, $\triangle CAK$ and $\triangle ABJ$ are clockwise oriented. Then we can apply Exercise 7 to deduce the identities

$$\ell = \frac{1-i}{2}c + \frac{1+i}{2}b,$$

$$k = \frac{1-i}{2}a + \frac{1+i}{2}c,$$

$$j = \frac{1-i}{2}b + \frac{1+i}{2}a.$$

The condition $\sphericalangle LJK = 90^\circ$ means that (see Exercise 6)

$$j = \frac{\lambda^2 - i\lambda}{1 + \lambda^2}\ell + \frac{1 + i\lambda}{1 + \lambda^2}k$$

where $\lambda > 0$ in case of counterclockwise orientation of $\triangle LKJ$ (Figure 4) and $\lambda < 0$ in case of clockwise orientation of $\triangle LKJ$ (Figure 5). Then one can combine all relations to obtain

$$a - \ell = \lambda(j - \ell),$$

so A lies on the line JL . In a similar way B lies on the line JK .

We leave as an exercise to the reader to prove the converse statement.

□

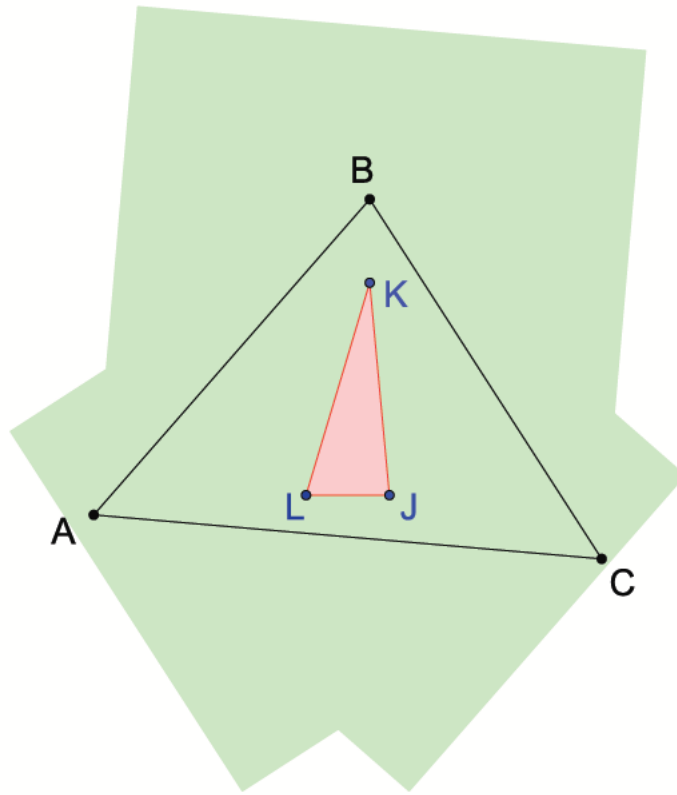


Figure 5: Right triangles with clockwise orientation of $\triangle LKJ$

3 Proof of Napoleon's theorem by using complex numbers

Now we are ready to complete the proof of Napoleon's theorem. Taking a look at Figure 1 and applying Lemma 3 we see that

$$\ell = w_1 b + w_2 c, \quad k = w_1 c + w_2 a, \quad j = w_1 a + w_2 b, \quad (3)$$

where

$$w_1 = \frac{z_1 + 1}{3}, \quad w_2 = \frac{z_2 + 1}{3}.$$

This simple relation guarantees that the centroids of $\triangle ANC$ and $\triangle LKJ$ coincide, since

$$\ell + j + k = a + b + c$$

in view of (10) and (3). Without loss of generality we may assume that

$$a + b + c = 0, \quad (4)$$

so

$$\ell + j + k = 0.$$

We have to verify that

$$\ell = z_1 j + z_2 k. \quad (5)$$

On one side Lemma 2 will guarantee (taking into account the orientation) that $\triangle LKJ$ is equilateral as stated in the Theorem. On the other hand, the substitution of ℓ, k, j from (3) in (5) leads to the relation

$$-(z_1 w_1 + z_2 w_2)a + (w_1 - z_1 w_2)b + (w_2 - z_2 w_1)c = 0. \quad (6)$$

Comparing this relation with (4), we see that the identities

$$-(z_1 w_1 + z_2 w_2) = (w_1 - z_1 w_2) = (w_2 - z_2 w_1)$$

will guarantee that (6) is satisfied. Having in mind that

$$w_1 = \frac{z_1 + 1}{3}, w_2 = \frac{z_2 + 1}{3}, z_1 + z_2 = 1,$$

we see that the following conditions have to be verified

$$-z_1^2 - z_2^2 - 1 = 1 - z_1 z_2 = 1 - z_1 z_2. \quad (7)$$

Now we can use (9) and see that

$$-z_1^2 - z_2^2 - 1 = z_2 + z_1 - 1 = 0, \quad 1 - z_1 z_2 = 1 - 1 = 0$$

so (7) is trivially satisfied and Napoleon's theorem is proved.

4 Some generalizations of Napoleon's theorem

A well - known generalization of Napoleon's theorem is the following one (see [12]):

For an arbitrary $\triangle ABC$, three exterior points A_1, B_1, C_1 are constructed such that

$$\triangle ABC_1 \sim \triangle BCA_1 \sim \triangle CAB_1.$$

Then the centroids of these triangles are vertices of a triangle similar to them.

Actually it's not even necessary to consider the centroids. For arbitrary $\triangle ABC$ take three external points A_1, B_1, C_1 such that

$$\sphericalangle AC_1 B + \sphericalangle BA_1 C + \sphericalangle CB_1 A = 360^0.$$

Then $\triangle A_1 B_1 C_1$ is similar to a triangle with angles

$$\sphericalangle C_1 A B + \sphericalangle B_1 A C, \sphericalangle C_1 B A + \sphericalangle A_1 B C, \sphericalangle A_1 C B + \sphericalangle B_1 C A.$$

One can see the references ([12], pp. 178 – 181) and [5] for the proof of this interesting fact.

A direct generalization of Napoleon's theorem was obtained first by Barlotti [1] in 1955 and then by Greber [6] in 1980. It says that if regular n -gons are erected outwardly(inwardly) on the sides of an n -gon P , then their centers are vertices of a regular n -gon if and only if P is affine-regular, i.e. it is the image of a regular n -gon under an affine transformation of the plane.

We propose to the reader to prove the particular case of the Theorem of Barlotti - Greber when $n = 4$.

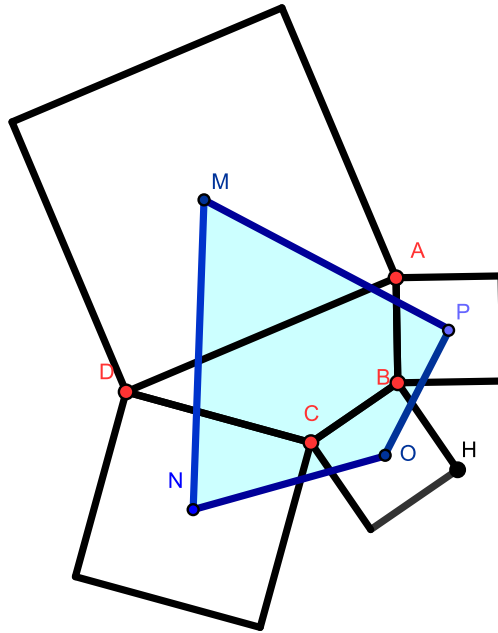


Figure 6: Geogebra application: Quadrilaterals and Napoleon's theorem

Exercise 2. *On the sides of a quadrilateral external squares are constructed. Prove that:*

(a) *The centers of the squares are vertices of a quadrilateral with perpendicular diagonals of equal length.*

(b) *The quadrilateral in (a) is a square if and only if the initial quadrilateral is a parallelogram.*

Hint. Let a, b, c, d be the complex numbers corresponding to the vertices of the original quadrilateral. Express the complex numbers of the centers M, N, P, Q of the squares in terms of a, b, c, d , and then show that $n - q = i(m - p)$. For part (b), use the fact that $MP \perp NQ$, showing that the quadrilateral $MNPQ$ is a square if and only if $MN \parallel PQ$. Then express this condition in terms of a, b, c, d .

Motivated by the Exercise 1 we propose to the reader to prove the following generalization of Napoleon's theorem:

Exercise 3. *On the sides of a non-equilateral triangle three regular n -gons are constructed externally to the triangle. Prove that their centers are vertices of an equilateral triangle iff $n = 3$.*

Hint. Use the fact (prove it) that the complex numbers a, b, c are vertices of an equilateral triangle iff

$$a^2 + b^2 + c^2 = ab + bc + ca.$$

5 Appendix: Complex numbers and geometry

One of the main difficulties for first year students at University is of course the math course (courses) and especially the lack of experience to work with trigonometric functions and complex numbers. The use of the complex numbers is not used effectively in the preparation of future teachers. Typical opinion is that this is an algebraic algorithm that is not very clear, but has to work by applying formal calculations only.

The initial observation is that any point (say A) in the plane can be identified with a complex number (denoted by a). If $a \in \mathbb{C}$ is multiplied by $\lambda > 0$ then we can interpret the map

$$a \in \mathbb{C} \Rightarrow \lambda a \in \mathbb{C}$$

as homothety or dilation. The multiplication by $e^{i\varphi} = \cos \varphi + i \sin \varphi$ for a real φ is a well defined map

$$a \in \mathbb{C} \Rightarrow e^{i\varphi} a \in \mathbb{C}$$

which is a rotation with center 0 and angle φ .

For any $\triangle ABC$ there is a relation of the form

$$a = z_1 b + z_2 c, \quad z_1 + z_2 = 1, \tag{8}$$

where $z_1, z_2 \in \mathbb{C}$.

It is clear that the coefficients z_1, z_2 are unique provided $B \neq C$. Indeed, if

$$z_1 b + z_2 c = \tilde{z}_1 b + \tilde{z}_2 c$$

and

$$z_1 + z_2 = \tilde{z}_1 + \tilde{z}_2 = 1$$

then we have

$$(z_1 - \tilde{z}_1)b = (\tilde{z}_2 - z_2)c.$$

Since $b \neq c$ and $z_1 - \tilde{z}_1 = \tilde{z}_2 - z_2$, we get

$$z_1 = \tilde{z}_1, z_2 = \tilde{z}_2.$$

We have the following

Lemma 2. $\triangle ABC$ is equilateral if and only if (8) is fulfilled with

$$z_1 = \frac{1}{2} \pm i \frac{\sqrt{3}}{2}, z_2 = \bar{z}_1.$$

Proof. We may assume that the complex number

$$m = \frac{b+c}{2}$$

corresponding to the midpoint of the segment BC is 0. Then

$$a = \pm i \tan(\pi/3)b = \pm i\sqrt{3}b,$$

since A can be obtained by a rotation by $\pi/2$ (i.e. by multiplication by $\pm i$) and homothety (i.e. multiplication) by $\tan(\pi/3) = \sqrt{3}$.

□

Remark 1. The numbers z_1, z_2 are the complex roots of the equation $z^3 = 1$ and moreover

$$z_1 + z_2 = 1, z_1 z_2 = 1, z_1^2 = -z_2, z_2^2 = -z_1. \quad (9)$$

Remark 2. The relation (8) is useful and can be easily adapted to more general situations and possible modifications of Napoleon's classical theorem.

Exercise 4. Find a necessary and sufficient condition (expressed in terms of z_1, z_2 in (8)) so that ABC is a right triangle with $\sphericalangle A = 90^\circ$.

Hint. The triangle ABC is right if and only if

$$c - a = (b - a)i\lambda$$

for some real $\lambda \neq 0$. Using this relation and (8) one finds

$$(z_1 - i\lambda z_1 + i\lambda)(c - b) = 0$$

which implies

$$z_1 - i\lambda z_1 + i\lambda = 0.$$

Answer.

$$z_1 = \frac{\lambda^2 - i\lambda}{1 + \lambda^2}, z_2 = 1 - z_1 = \frac{1 + i\lambda}{1 + \lambda^2}$$

for some real number $\lambda \neq 0$.

Exercise 5. Find a necessary and sufficient condition (expressed in terms of z_1, z_2 in (8)) so that ABC is an isosceles right triangle with $\sphericalangle A = 90^\circ$.

Hint. Take $\lambda = \pm 1$ in Exercise 4 to obtain

$$z_1 = \frac{1 \mp i}{2}, z_2 = \overline{z_1}.$$

The following Lemma has been used in the proof of Napoleon's theorem.

Lemma 3. If $\triangle ABC$ is equilateral and (8) is fulfilled with

$$z_1 = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}, z_2 = \overline{z_1},$$

then its centroid is given by

$$\frac{a + b + c}{3} = w_1 b + w_2 c,$$

where

$$w_1 = \frac{(z_1 + 1)}{3}, w_2 = \frac{(z_2 + 1)}{3}.$$

Note that (9), implies that

$$w_1 + w_2 = 1. \quad (10)$$

We end this section giving a more precise description of the choice of the points L, K, J in Figure 1. The orientation of a circle in the plane is clockwise or counterclockwise orientation. Similarly, any three points or any triangle has clockwise or counterclockwise orientation. The triangle ABC on Figure 1 has a clockwise orientation. It is important that $\triangle ABJ$, $\triangle BCL$ and $\triangle CAK$ have the same orientation.

We have the following modification of Lemma 2, where the orientation is taken into account.

Lemma 4. $\triangle ABC$ is equilateral and counterclockwise oriented iff (8) is fulfilled with

$$z_1 = \frac{1}{2} - i\frac{\sqrt{3}}{2}, z_2 = \bar{z}_1.$$

Similarly, Exercises 4 and 5 become

Exercise 6. A necessary and sufficient condition so that ABC is a right triangle with $\angle A = 90^\circ$ and counterclockwise oriented is

$$a = z_1 b + z_2 c,$$

where

$$z_1 = \frac{\lambda^2 - i\lambda}{1 + \lambda^2}, z_2 = 1 - z_1 = \frac{1 + i\lambda}{1 + \lambda^2}$$

for some real number $\lambda > 0$.

Remark 3. The condition

$$a = \frac{\lambda^2 - i\lambda}{1 + \lambda^2} b + \frac{1 + i\lambda}{1 + \lambda^2} c$$

can be rewritten in the form

$$a = \frac{1 + i\mu}{1 + \mu^2} b + \frac{\mu^2 - i\mu}{1 + \mu^2} c$$

after the substitution

$$\mu = -\frac{1}{\lambda}.$$

Exercise 7. A necessary and sufficient condition so that $\triangle ABC$ is isosceles right triangle with $\angle A = 90^\circ$ and counterclockwise oriented is

$$a = \frac{1 - i}{2} b + \frac{1 + i}{2} c.$$

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