Poncelet's porism and periodic triangles in ellipse

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1 Small historical introduction

One of the most important and beautiful theorems in projective geometry is that of Poncelet, concerning closed polygons which are inscribed in one conic and circumscribed about another (below we give the precise statement as well proof for the case of triangles). The theorem has deep interaction with other math fields. The aim of this section is to clarify one aspect of these these relations: the connection between Poncelet's theorem and billiards in an ellipse. At first sight these topics seem unrelated, belonging to two distinct mathematical fields: geometry and dynamical systems. But there is a hidden thread tying these topics together: the existence of an underlying structure (we name it the Poncelet correspondence which turns out to be an elliptic curve. As is well known, elliptic curves can be endowed with a group structure, and the exploitation of this structure sheds much light on the aforementioned topics.

However, to read most of the books and available references some prerequisites (usually covered in undergraduate and first year graduate mathematics courses) are needed: complex analysis, linear algebra, and some point set topology.

In this sense the argument can not be adapted easily to some extracurricula activities in High Schools.

For this we are trying to find approach that needs only tools from the standard High School Programs.

This is not an easy problem. The classical A. Cayley (see [2], [3]) approach uses elliptic integrals, some other sources (see [5], [6], [8] and the references cited there) apply arguments for projective geometry and group theory.

The statement of the Poncelet's problem needs only to know the definition and the equation of the ellipse.

Theorem 1. *(Poncelet's Porism) Given one ellipse inside another, if there exists one circum-inscribed (simultaneously inscribed in the outer and circumscribed on the inner) n -gon, then any point on the boundary of the outer ellipse is the vertex of some circuminscribed n-gon.*

There are several proofs of this remarkable theorem, most of which are not elementary. Poncelet's theorem dates to the nineteenth century and has attracted the attention of many mathematicians of that period (a detailed historical account is given in [1]). The main reason for this interest seems to stem from the fact that several proofs of this theorem require the use of complex and homogeneous coordinates, notions which were beginning to emerge at the time (1813) when Poncelet discovered his theorem. Poncelet discovered the theorem while in captivity as war prisoner in the Russian city of Saratov. After his return to France, a proof appears in his book [7], published in 1822. The proof, which is synthetic and somewhat elaborate, reduces the theorem to two (not' necessarily concentric) circles. A discussion of the ideas in Poncelet's proof is given in [1], pp. 298-311.

Our purpose is to find elementary proof in one nontrivial situation: the case $n = 3$ and the situation, when we have two ellipses

$$e : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

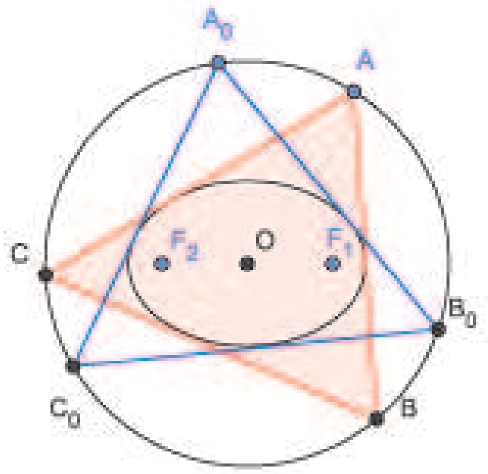

Figure 1: Poncelet's theorem for the case of circle and ellipse.

and

$$e_1: \ \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1, \tag{2}$$

such that e_1 is inside e.

We shall prove in this case the Poncelet' theorem as well as the following more precise result.

Theorem 2. *(see Figure 1) Suppose the ellipse* (2) *is inside the ellipse* (1)*, i.e.*

$$a > b > 0, a_1 > b_1 > 0,$$

$$a > a_1, b > b_1.$$

Then the following conditions are equivalent:

i) there exists a triangle $\triangle A_0B_0C_0$ *inscribed in* e *and circumscribed on* e_1*,*

ii) we have the relation

$$\frac{a_1}{a} + \frac{b_1}{b} = 1.$$

iii) for any point A *on the ellipse* e *one can find a unique triangle* $\triangle ABC$ *inscribed in* e *and circumscribed on* e_1*.*

2 Reduction to the case of circle and ellipse and preliminary facts

Consider two ellipses

$$e: \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{3}$$

and

$$e_1: \ \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1, \tag{4}$$

such that e_1 is inside e. This condition can be expressed as

$$a > b > 0, a_1 > b_1 > 0,$$

$$a > a_1, b > b_1.$$

One can use a simple change of coordinates in the plane

$$X = \frac{x}{a}, \quad Y = \frac{y}{b}, \tag{5}$$

so that the ellipse e in the new coordinates X, Y has equation

$$X^2 + Y^2 = 1. \tag{6}$$

so it is the circle $k(O, 1)$ with center at the origin O of the new coordinate system and has radius 1.

The second ellipse e_1 becomes

$$\frac{X^2}{A_1^2} + \frac{Y^2}{B_1^2} = 1, \quad A_1 = \frac{a_1}{a}, B_1 = \frac{b_1}{b} \tag{7}$$

and it is clear that this change of coordinates preserves the notions of intersection, line is transformed in line, circle in circle, ellipse in ellipse (or circle as a partial case) and if the line and ellipse are tangent they remain tangent after the change of the coordinates (see Figure 2).

Exercise 1. *Prove the fact that if line and ellipse are tangent they remain tangent after the change of the coordinates* (5).

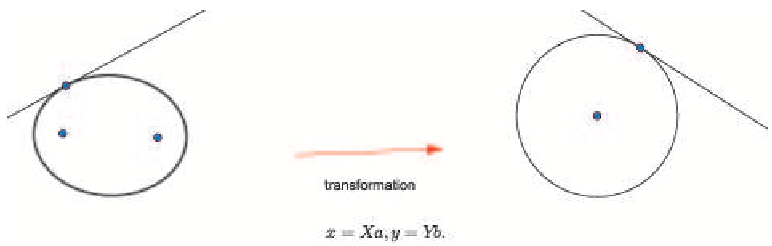

Figure 2: Ellipse is transformed in circle.

For this from now on we shall work with circle $k(O, 1)$ with center at the origin O and radius 1

$$x^2 + y^2 = 1. \tag{8}$$

and ellipse e_1

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1, \quad 1 > a_1 \geq b_1 \tag{9}$$

inside $k(O, 1)$ as it is shown on Figure 1.

We prepare again a list of questions preparing the solution of the problem (or proof of the Poncelet's theorem):

- Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(x_0, y_0)$ on $k(O, 1)$ find the tangent lines from A_0 to e_1 and find also the points A_1, A_2 of the intersection of these tangent lines with the circle $x^2 + y^2 = 1$ (we need formula expressing the coordinates of A_1, A_2 in terms of x_0, y_0 and the angular coefficients k_1, k_2 of the lines A_0A_1 and A_0A_2 respectively;

- Using the parametrization

$$x_j = \cos\varphi_j, y_j = \sin\varphi_j,\ j = 0,1,2 \tag{10}$$

  find a relation between φ_j and $\theta_{1,2} = arctank_{1,2}$.

- Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$, the point $A_0(x_0, y_0)$ on $k(O, 1)$, the tangent lines from A_0 to e_1 intersecting $k(O, 1)$ into the points A_1, A_2 and using the parametrization (10) express the necessary and sufficient condition that the line A_0A_1 is tangent to the ellipse e_1 in terms of φ_0, φ_1 and $\theta_1 = arctank_1$.

- Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$, the point $A_0(x_0, y_0)$ on $k(O, 1)$, the tangent lines from A_0 to e_1 intersecting $k(O, 1)$ into the points A_1, A_2 and using the parametrization (10) express the necessary and sufficient condition that the line A_0A_2 is tangent to the ellipse e_1 in terms of φ_0, φ_2 and $\theta_2 = arctank_2$.

- Using simple trigonometric transformations show that the following two conditions a) the line A_0A_1 is tangent to the ellipse e_1 (condition is expressed in terms of φ_0, φ_1 and $\theta_1 = arctank_1$) b) the line A_0A_2 is tangent to the ellipse e_1 (condition is expressed in terms of φ_0, φ_2 and $\theta_2 = arctank_2$) imply a) the line A_1A_2 is tangent to the ellipse e_1 (condition is expressed in terms of φ_1, φ_2 and $\theta_{1,2} = arctank_{1,2}$)

Step by step we give answers presenting some Lemmas that can be verified without difficulty.

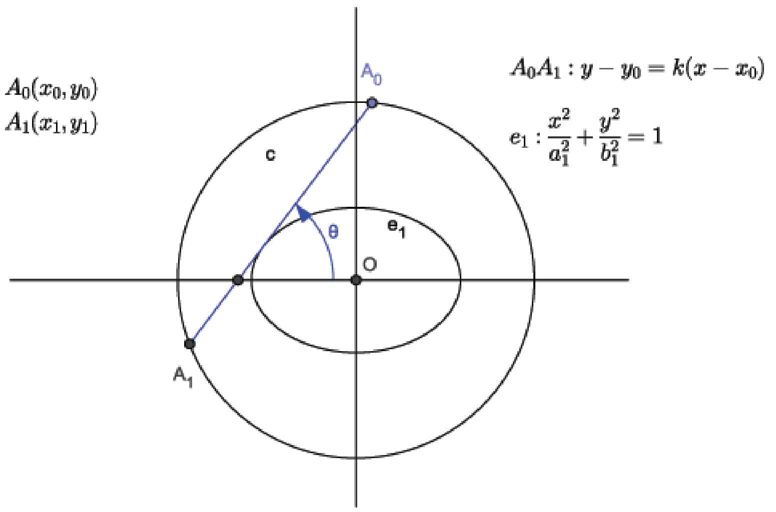

Figure 3: When A_0A_1 is tangent to e_1?.

Lemma 1. *Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ one can express the necessary and sufficient condition such that the line $y - y_0 = k(x - x_0)$ through the point $A_0(x_0, y_0)$ is tangent to e_1 as follows*

$$(y_0 - kx_0)^2 = b_1^2 + k^2a_1^2.$$

Lemma 2. *Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and point $A_0(x_0, y_0)$ on the unit circle and denote by*

$$t : y - y_0 = k(x - x_0)$$

any line through A_0 *and by* $A_1(x_1, y_1)$ *the point of the second intersection of this line with the unit circle* $k(O, 1) : x^2 + y^2 = 1$, *such we have*

$$x_1 = \frac{k^2 - 1}{k^2 + 1} x_0 - \frac{2k}{k^2 + 1} y_0,$$

$$y_1 = -\frac{2k}{k^2 + 1} x_0 - \frac{k^2 - 1}{k^2 + 1} y_0.$$

Proof. The intersection points are given by the equations

$$x^2 + (y_0 + k(x - x_0))^2 = 1.$$

This equation has two roots x_0 and x_1 so

$$x_0 + x_1 = -\frac{2k(y_0 - kx_0)}{1 + k^2}.$$

From this relation we get the expression for x_1. Similarly we proceed for y_1. □

Lemma 3. *Given an ellipse* $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ *and a point* $A_0(\cos\varphi_0, \sin\varphi_0)$ *on the unit circle denote by*

$$t : y - y_0 = k(x - x_0)$$

any line from A_0 *and let* A_1 *the second point of intersection of this lines with the circle* $k(O, 1) : x^2 + y^2 = 1$, *such that* $A_1(\cos\varphi, \sin\varphi)$. *Then the relations of Lemma 2 take the form we have*

$$\theta = \frac{\varphi + \varphi_0 - \pi}{2} + m\pi, m \in \mathbb{Z},$$

where

$$\theta = \arctan k.$$

Proof. We have the relations

$$\frac{k^2 - 1}{k^2 + 1} = -\cos(2\theta), \quad \frac{2k}{k^2 + 1} = \sin(2\theta).$$

Making the substitution

$$x_1 = \cos\varphi, y_1 = \sin\varphi$$

we find

$$\cos\varphi = -\cos(2\theta)\cos\varphi_0 - \sin(2\theta)\sin\varphi_0 =$$
$$= \cos(2\theta + \pi)\cos\varphi_0 + \sin(2\theta + \pi)\sin\varphi_0 = \cos(2\theta + \pi - \varphi_0),$$
$$\sin\varphi = -\sin(2\theta)\cos\varphi_0 + \cos(2\theta)\sin\varphi_0 =$$
$$= \sin(2\theta + \pi)\cos\varphi_0 - \cos(2\theta + \pi)\sin\varphi_0 = \sin(2\theta + \pi - \varphi_0),$$

and these relations lead simply to the needed relation

$$2\theta + \pi - \varphi_0 = \varphi + 2m\pi, m \in \mathbb{Z}.$$

This completes the proof.

□

Lemma 4. *Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and a point $A_0(\cos\varphi_0, \sin\varphi_0)$ denote by*

$$t : y - y_0 = k(x - x_0)$$

a line through A_0 and by A_1 the point of the intersection of this line with the circle $e : x^2+y^2 = 1$, such that $A_1(\cos\varphi, \sin\varphi)$. Then t is tangent to e_1 if and only if

we have

$$\cos^2\left(\frac{\varphi - \varphi_0}{2}\right) = b_1^2 \sin^2\left(\frac{\varphi + \varphi_0}{2}\right) + a_1^2 \cos^2\left(\frac{\varphi + \varphi_0}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\varphi + \varphi_0}{2}\right) + b_1^2.$$

Proof. From Lemma 1 we see that we need to transform $(y_0 - kx_0)^2$ into a function of φ and φ_0. Indeed, we have

$$y_0 - kx_0 = \frac{\cos\theta \sin\varphi_0 - \sin\theta\cos\varphi_0}{\cos\theta} = \frac{\sin(\varphi_0 - \theta)}{\cos\theta}. \tag{11}$$

Using now the relation

$$\theta = \frac{\varphi + \varphi_0 - \pi}{2} + m\pi, m \in \mathbb{Z},$$

from Lemma 1, we see the the numerator in (11) is

$$\sin(\varphi_0 - \theta) = \sin\left(\frac{\varphi_0 - \varphi + \pi}{2} - m\pi\right) = (-1)^m \cos\left(\frac{\varphi_0 - \varphi}{2}\right)$$

while the denominator becomes

$$\cos\theta = \cos\left(\frac{\varphi + \varphi_0 - \pi}{2} + m\pi\right) = (-1)^m \sin\left(\frac{\varphi + \varphi_0}{2}\right)$$

so we find

$$\sin^2\left(\frac{\varphi + \varphi_0}{2}\right)(y_0 - kx_0)^2 = \cos^2\left(\frac{\varphi - \varphi_0}{2}\right).$$

Applying Lemma 1 combined with the above relations, we complete the proof of the Lemma.

□

Remark 1. *We can rewrite the relations of Lemma 4 in different ways using the formula*

$$\cos^2\alpha = \frac{1 + \cos(2\alpha)}{2},$$

also as

$$\cos(\varphi - \varphi_0) = c^2 \cos(\varphi + \varphi_0) + D, \tag{12}$$

or

$$(1 - c^2)\cos\varphi\cos\varphi_0 + (1 + c^2)\sin\varphi\sin\varphi_0 = D, \tag{13}$$

where

$$c^2 = a_1^2 - b_1^2, D = a_1^2 + b_1^2 - 1. \tag{14}$$

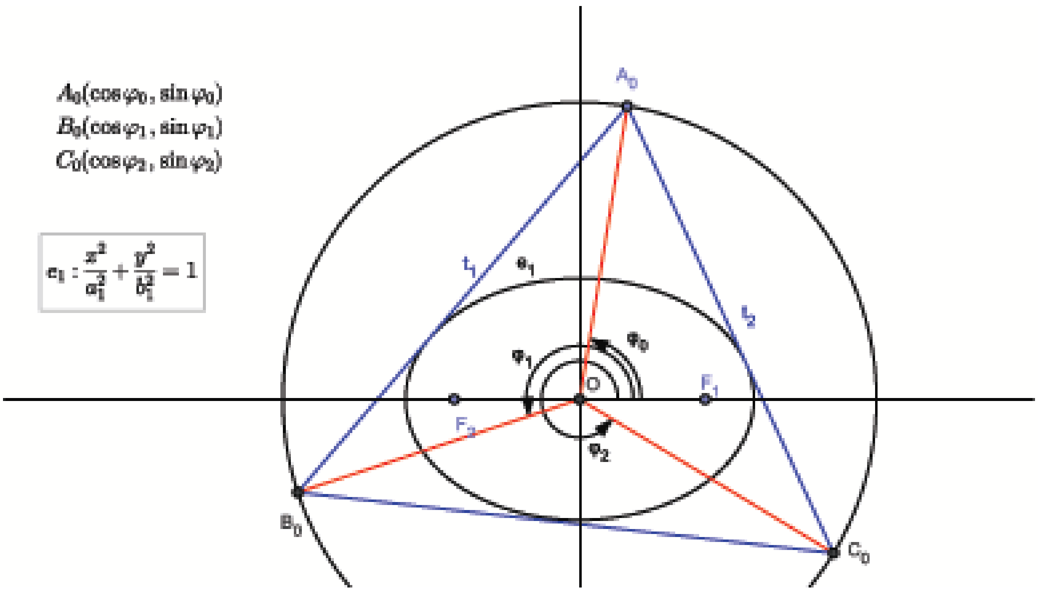

Figure 4: The meaning of the assumption $\triangle\, A_0B_0C_0$ is circumscribed on e_1?

3 Proof of Poncelet theorem using trigonometric functions

We take a point $A_0(cos\varphi_0\,, \sin\varphi_0)$ on the unit circle and find of two tangent lines t_1, t_2 through A_0 to the ellipse

$$e_1 = \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1.$$

Then we find the intersection points of t_1, t_2 with the unit circle (see Figure 4) and denote the two intersection points (different from A_0) by

$$B_0(cos\varphi_1\,, \sin\varphi_1), C_0(cos\varphi_2\,, \sin\varphi_2).$$

First, let us express the assumption of Poncelet's theorem that there exists at least one triangle $\triangle\, A_0B_0C_0$ inscribed in the unit circle, i.e.

$$A_0(cos\varphi_0\,, \sin\varphi_0), B_0(cos\varphi_1\,, \sin\varphi_1), C_0(cos\varphi_2\,, \sin\varphi_2), \quad 0 \le \varphi_0 < \varphi_1 < \varphi_2 \le 2\pi$$

and circumscribed on the inner ellipse e_1 Since A_0B_0 is tangent to e_1 we know that:

$$\cos^2\left(\frac{\varphi_1 - \varphi_0}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\varphi_1 + \varphi_0}{2}\right) + b_1^2 \tag{15}$$

(this is due to Lemma 4). Similarly, the fact that A_0C_0 and B_0C_0 are tangent to e_1, and Lemma 4 imply

$$\cos^2\left(\frac{\varphi_2 - \varphi_0}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\varphi_2 + \varphi_0}{2}\right) + b_1^2. \tag{16}$$

$$\cos^2\left(\frac{\varphi_2 - \varphi_1}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\varphi_2 + \varphi_1}{2}\right) + b_1^2. \tag{17}$$

We can unify all these relations into one

$$\cos^2\left(\frac{\varphi_j - \varphi_\ell}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\varphi_j + \varphi_\ell}{2}\right) + b_1^2, \quad 0 \le j \ne \ell \le 2. \tag{18}$$

What we know from the assumptions the Poncelet theorem and what we have to prove?

Take any point $A(cos\psi_0\,,\sin\psi_0)$ on the unit circle and find of two tangent lines t_1, t_2 through A_0 to the ellipse

$$e_1 : \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1.$$

Then we find the intersection points of t_1, t_2 with the unit circle (see Figure 5) and denote the two intersection points (different from A) by

$$B(cos\psi_1\,,\sin\psi_1), C(cos\psi_2\,,\sin\psi_2).$$

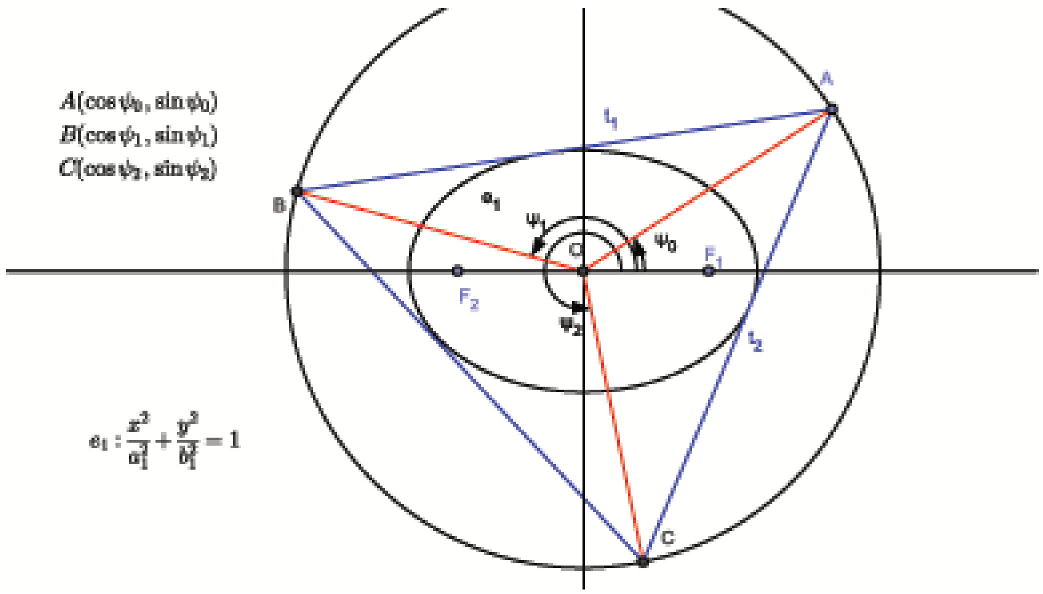

Figure 5: Two sides tangent $\Rightarrow$ the third side is also tangent.

Since AB is tangent to e_1 we know that:

$$\cos^2\left(\frac{\psi_1 - \psi_0}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\psi_1 + \psi_0}{2}\right) + b_1^2 \tag{19}$$

(this is due to Lemma 4). Similarly, the fact that A_0C_0 and B_0C_0 are tangent to e_1, and Lemma 4 imply

$$\cos^2\left(\frac{\psi_2 - \psi_0}{2}\right) + (a_1^2 - b_1^2)\cos^2\left(\frac{\psi_2 + \psi_0}{2}\right) = b_1^2. \tag{20}$$

So we summarize all assumptions of Poncelet's theorem and can say that (18), (19) and (20) are satisfied.

What we have to prove?

Having in ming again Lemma 4 we see that our purpose is to show that

$$\cos^2\left(\frac{\psi_2 - \psi_1}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\psi_2 + \psi_1}{2}\right) + b_1^2. \tag{21}$$

This relation can be rewritten as

$$(1-c^2)\cos\psi_2\cos\psi_1 = (1+c^2)\sin\psi_2\sin\psi_1 + D, \tag{22}$$

where

$$c^2 = a_1^2 - b_1^2, D = a_1^2 + b_1^2 - 1. \tag{23}$$

according to Remark 1.

Now we are in position to apply the trigonometric lemma from the appendix and conclude that

$$\cos^2\left(\frac{\psi_2-\psi_1}{2}\right) = \frac{4c^2D^2}{(1-c^2)^2(1+c^2)^2}\cos^2\left(\frac{\psi_2+\psi_1}{2}\right) + \frac{D^2}{(1+c^2)^2}. \tag{24}$$

Comparing this relation with (21) we see that the following conditions

$$4D^2 = (1-c^2)^2(1+c^2)^2,\ D^2 = b_1^2(1+c^2)^2 \tag{25}$$

are required. This relations and (23) lead to the following sufficient condition

$$a_1 + b_1 = 1 \tag{26}$$

that implies $\triangle\ ABC$ is circumscribed on e_1. The condition (23) is also necessary for the fulfillment of the property

- there exists a triangle $\triangle\ A_0B_0C_0$ circumscribed on e_1.

If there exists at least one $\triangle\ A_0B_0C_0$ circumscribed on e_1, then (26) and hence $\triangle\ ABC$ is circumscribed on e_1.

This completes the proof of the Theorem.

4 Appendix: Trigonometric Lemma

Lemma 5. *Suppose*

$$\sin\left(\frac{\psi_1-\psi_2}{2}\right) \neq 0,\ \cos\left(\frac{\psi_1+\psi_2}{2}\right) \neq 0, \cos\psi_0$$

and

$$\begin{cases} (1-c^2)\cos\psi_1\cos\psi_0 + (1+c^2)\sin\psi_1\sin\psi_0 = D & ; \\ (1-c^2)\cos\psi_2\cos\psi_0 + (1+c^2)\sin\psi_2\sin\psi_0 = D & . \end{cases} \tag{27}$$

Then

$$(1-c^2)\tan\left(\frac{\psi_1+\psi_2}{2}\right) = (1+c^2)\tan\psi_0 \tag{28}$$

and moreover

$$\cos^2\left(\frac{\psi_2-\psi_1}{2}\right) = \frac{4c^2D^2}{(1-c^2)^2(1+c^2)^2}\cos^2\left(\frac{\psi_2+\psi_1}{2}\right) + \frac{D^2}{(1+c^2)^2}. \tag{29}$$

Proof. Take the difference between the relations in (27). We get

$$-(1-c^2)\sin\left(\frac{\psi_1-\psi_2}{2}\right)\sin\left(\frac{\psi_1+\psi_2}{2}\right)\cos\psi_0+(1+c^2)\sin\left(\frac{\psi_1-\psi_2}{2}\right)\cos\left(\frac{\psi_1+\psi_2}{2}\right)\sin\psi_0=0.$$

The assumption

$$\sin\left(\frac{\psi_1-\psi_2}{2}\right)\neq 0$$

implies that

$$(1-c^2)\sin\left(\frac{\psi_1+\psi_2}{2}\right)\cos\psi_0=(1+c^2)\cos\left(\frac{\psi_1+\psi_2}{2}\right)\sin\psi_0.$$

This proves (28). The other relation can be obtained following the plan

- first equation in (27) $\times\sin\psi_2-$ second equation in (27) $\times\sin\psi_1$;
- first equation in (27) $\times\cos\psi_2-$ second equation in (27) $\times\cos\psi_1$.

In this way we get

$$2D\sin\left(\frac{\psi_2-\psi_1}{2}\right)\cos\left(\frac{\psi_2+\psi_1}{2}\right)=2(1-c^2)\sin\left(\frac{\psi_2-\psi_1}{2}\right)\cos\left(\frac{\psi_2-\psi_1}{2}\right)\cos\psi_0,$$

$$-2D\sin\left(\frac{\psi_2-\psi_1}{2}\right)\sin\left(\frac{\psi_2+\psi_1}{2}\right)=-2(1+c^2)\sin\left(\frac{\psi_2-\psi_1}{2}\right)\cos\left(\frac{\psi_2-\psi_1}{2}\right)\sin\psi_0,$$

so using the assumption

$$\sin\left(\frac{\psi_1-\psi_2}{2}\right)\neq 0$$

we find

$$\frac{D}{1-c^2}\cos\left(\frac{\psi_2+\psi_1}{2}\right)=\cos\left(\frac{\psi_2-\psi_1}{2}\right)\cos\psi_0,$$

$$\frac{D}{1+c^2}\sin\left(\frac{\psi_2+\psi_1}{2}\right)=\cos\left(\frac{\psi_2-\psi_1}{2}\right)\sin\psi_0.$$

Taking the sum of squares of these identities we obtain

$$\frac{D^2}{(1-c^2)^2}\cos^2\left(\frac{\psi_2+\psi_1}{2}\right)+\frac{D^2}{(1+c^2)^2}\sin^2\left(\frac{\psi_2+\psi_1}{2}\right)=\cos^2\left(\frac{\psi_2-\psi_1}{2}\right)$$

and this equation yields (29).

This completes the proof of the Lemma.

□

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