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Poncelet's porism and periodic triangles in ellipse Modelling optical lenses with Dynamic Geometry Software Andreas Ulovec

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1 Small historical introduction

One of the most important and beautiful theorems in projective geometry is that of Poncelet, concerning closed polygons which are inscribed in one conic and circumscribed about another (below we give the precise statement as well proof for the case of triangles). The theorem has deep interaction with other math fields. The aim of this section is to clarify one aspect of these these relations: the connection between Poncelet's theorem and billiards in an ellipse. At first sight these topics seem unrelated, belonging to two distinct mathematical fields: geometry and dynamical systems. But there is a hidden thread tying these topics together: the existence of an underlying structure (we name it the Poncelet correspondence which turns out to be an of an underlying structure (we have a lens reflected correspondence with a group structure, and the elliptic curve. As is well known, elliptic curves can be endowed with a group structure, and the exploitation of this structure sheds much light on the aforementioned topics. Due of the most important and beautiful theorems in projective geometry is that of ronce

However, to read most of the books and available references some prerequisites (usually covered in undergraduate and first year graduate mathematics courses) are needed: complex analysis, linear algebra, and some point set topology. reflection and refraction – not *instead* of the actual experiment (if one sees experiments only in However, to read most of the books and available references some prerequisites (usually cover

In this sense the argument can not be adapted easily to some extracurricula activities in High Schools. another part penetrates the glass and continues there, in another angle. The same happens when the light reaches the original can not be adapted easily to some extracurricula activities in $\boldsymbol{\Pi}$ δ chools.

For this we are trying to find approach that needs only tools from the standard High School Programs. near the centre of the centre of the lenses and light being more off-centre. We calculate the control of the c become more complex, and from the equations alone it would be different to see what happens. With α

This is not an easy problem. The classical A. Cayley (see $[2]$, $[3]$) approach uses elliptic integrals, some other sources (see $[5]$, $[6]$, $[8]$ and the references cited there) apply arguments for projective geometry and group theory.

The statement of the Poncelet's problem needs only to know the definition and the equation of the ellipse. **2 Easy beginnings – light hits a plane surface**

Theorem 1. (Poncelet's Porism) Given one ellipse inside another, if there exists one circuminscribed (simultaneously inscribed in the outer and circumscribed on the *inner*) n -gon, then any point on the boundary of the outer ellipse is the vertex of some circuminscribed n-gon.

There are several proofs of this remarkable theorem, most of which are not elementary. Poncelet's theorem dates to the nineteenth century and has attracted the attention of many mathematicians of that period (a detailed historical account is given in [\[1\]](#page-10-5)). The main reason for this interest seems to stem from the fact that several proofs of this theorem require the use of complex and homogeneous coordinates, notions which were beginning to emerge at the time (1813) when Poncelet discovered his theorem. Poncelet discovered the theorem while in captivity as war prisoner in the Russian city of Saratov. After his return to France, a proof appears in his book [\[7\]](#page-10-6), published in 1822. The proof, which is synthetic and somewhat elaborate, reduces the theorem to two (not' necessarily concentric) circles. A discussion of the ideas in Poncelet's proof is given in [\[1\]](#page-10-5), pp. 298-311.

Our purpose is to find elementary proof in one nontrivial situation: the case $n = 3$ and the situation, when we have two ellipses

$$
e: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}
$$

Figure 1: Poncelet's theorem for the case of circle and ellipse. for mathematics the mathematics teachers. The mathematic mathematics is the mathematical ray of it in the mathematics.

and and continues the glass and continues the glass and continues the same happens when the s

$$
e_1: \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1,\tag{2}
$$

such that e_1 is inside e .

We shall prove in this case the Poncelet' theorem as well as the following more precise result. \mathbf{D} is possible to simulate the properties of a lens without actually having to use a lens, laser light, laser l

Theorem 2. (see Figure [1](#page-1-0)) Suppose the ellipse (2) is inside the ellipse (1) , i.e.

$$
a > b > 0, a_1 > b_1 > 0,
$$

$$
a>a_1, b>b_1.
$$

Then the following conditions are equivalent:

- i) there exists a triangle $\Delta A_0 B_0 C_0$ inscribed in e and circumscribed on e_1 ,
- ii) we have the relation

$$
\frac{a_1}{a} + \frac{b_1}{b} = 1.
$$

iii) for any point A on the ellipse e one can find a unique triangle \triangle ABC inscribed in e and circumscribed on e1.

2 Reduction to the case of circle and ellipse and preliminary facts

Consider two ellipses

$$
e: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
$$
\n(3)

and

$$
e_1: \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1,\tag{4}
$$

such that e_1 is inside e . This condition can be expressed as

$$
a > b > 0, a_1 > b_1 > 0,
$$

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$$
a>a_1, b>b_1.
$$

One can use a simple change of coordinates in the plane Andreas Ulovec

$$
X = \frac{x}{a}, \quad Y = \frac{y}{b},\tag{5}
$$

so that the ellipse e in the new coordinates X, Y has equation

$$
X^2 + Y^2 = 1.
$$
 (6)

so it is the circle $k(O, 1)$ with center at the origin O of the new coordinate system and has radius 1. thicker, you have to take out the current lens and put in the new one. Students can then observe the

The second ellipse e_1 becomes

$$
\frac{X^2}{A_1^2} + \frac{Y^2}{B_1^2} = 1, \quad A_1 = \frac{a_1}{a}, B_1 = \frac{b_1}{b} \tag{7}
$$

and it is clear that this change of coordinates preserves the notions of intersection, line is transformed in line, circle in circle, ellipse in ellipse (or circle as a partial case) and if the line and ellipse are tangent they remain tangent after the change of the coordinates (see Figure [2\)](#page-2-0). and empse are part perfective and continues the glass and continues (see Figure.

Exercise 1. Prove the fact that if line and ellipse are tangent they remain tangent after the $change\ of\ the\ coordinates\ (5).$ $change\ of\ the\ coordinates\ (5).$ $change\ of\ the\ coordinates\ (5).$ \mathcal{L}_{F} and coordinates Θ .

Figure 2: Ellipse is transformed in circle.

For this from now on we shall work with circle $k(O, 1)$ with center at the origin O and radius 1

$$
x^2 + y^2 = 1.
$$
 (8)

and ellipse e_1

$$
\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1, \quad 1 > a_1 \ge b_1 \tag{9}
$$

inside $k(O, 1)$ as it is shown on Figure [1.](#page-1-0)

We prepare again a list of questions preparing the solution of the problem (or proof of the Poncelet's theorem):

• Given an ellipse e_1 : $x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(x_0, y_0)$ on $k(O, 1)$ find the tangent lines from A_0 to e_1 and find also the points A_1, A_2 of the intersection of these tangent lines with the circle $x^2 + y^2 = 1$ (we need formula expressing the coordinates of A_1, A_2 in terms of x_0, y_0 and the angular coefficients k_1, k_2 of the lines A_0A_1 and A_0A_2 respectively;

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• Using the parametrization Andreas Ulovec

$$
x_j = \cos \varphi_j, y_j = \sin \varphi_j, \ j = 0, 1, 2 \tag{10}
$$

find a relation between φ_j and $\theta_{1,2} = arctank_{1,2}$. **1 1 1 1 1 1 1 1**

- Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$, the point $A_0(x_0, y_0)$ on $k(O, 1)$, the tangent lines from A_0 to e_1 intersecting $k(O, 1)$ into the points A_1, A_2 and using the parametrization [\(10\)](#page-3-0) express the necessary and sufficient condition that the line A_0A_1 is tangent to the ellipse e_1 in terms of φ_0, φ_1 and $\theta_1 = arctank_1$. \mathcal{L} optics when it comes down to show the path of rays of \mathcal{L} • Given an ellipse $e_1: x^2/a_1^2 + y^2/b_1^2 = 1$, the point $A_0(x_0, y_0)$ on $k(O, 1)$, the tangent list done by removing one piece and putting another piece in. To see what happens if you make a lens
- Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$, the point $A_0(x_0, y_0)$ on $k(O, 1)$, the tangent lines from A_0 to e_1 intersecting $k(O, 1)$ into the points A_1, A_2 and using the parametrization [\(10\)](#page-3-0) express the necessary and sufficient condition that the line A_0A_2 is tangent to the ellipse e_1 in terms of φ_0, φ_2 and $\theta_2 = arctank_2$. • Given an ellipse $e_1: x^2/a_1^2 + y^2/b_1^2 = 1$, the point $A_0(x_0, y_0)$ on $k(O, 1)$, the tangent li
- Using simple trigonometric transformations show that the following two conditions a) the line A_0A_1 is tangent to the ellipse e_1 (condition is expressed in terms of φ_0, φ_1 and $\theta_1 = arctank_1$) b) the line A_0A_2 is tangent to the ellipse e_1 (condition is expressed in terms of φ_0, φ_2 and $\theta_2 = arctank_2$ imply a) the line A_1A_2 is tangent to the ellipse e_1 (condition is expressed in terms of φ_1, φ_2 and $\theta_{1,2} = arctank_{1,2}$) T , who can be useful for science teachers, who can use it to model experiments with lenses, with lens • Using simple trigonometric transformations show that the following two conditions \mathcal{L} reaches the other surface of the lens – again mathematics is required to calculate the angle in \mathcal{L}

Step by step we give answers presenting some Lemmas that can be verified without difficulty. which the light is reflected and refracted. For ideal lenses, there is an easy equation calculating these ϵ but the push vertex well and it is seen and it does not with the vertex without difficult fall ϵ

Figure 3: When A_0A_1 is tangent to e_1 ?.

Lemma 1. Given an ellipse e_1 : $x^2/a_1^2 + y^2/b_1^2 = 1$ one can express the necessary and sufficient condition such that the line $y - y_0 = k(x - x_0)$ through the point $A_0(x_0, y_0)$ is tangent to e_1 as follows

$$
(y_0 - kx_0)^2 = b_1^2 + k^2 a_1^2.
$$

Lemma 2. Given an ellipse e_1 : $x^2/a_1^2 + y^2/b_1^2 = 1$ and point $A_0(x_0, y_0)$ on the unit circle and denote by **Fig.1** Reflection of light at a plane surface.

$$
t: y - y_0 = k(x - x_0)
$$

any line through A_0 and by $A_1(x_1, y_1)$ the point of the second intersection of this line with the unit circle $k(O, 1) : x^2 + y^2 = 1$, such we have **Modelling optical lenses with Dynamic Geometry Software**

$$
x_1 = \frac{k^2 - 1}{k^2 + 1} x_0 - \frac{2k}{k^2 + 1} y_0,
$$

$$
y_1 = -\frac{2k}{k^2 + 1} x_0 - \frac{k^2 - 1}{k^2 + 1} y_0.
$$

Proof. The intersection points are given by the equations dong. The intersection points are given by the equations

$$
x^2 + (y_0 + k(x - x_0))^2 = 1.
$$

This equation has two roots x_0 and x_1 so $\lim_{\omega \to 0}$ equation has two roots ω_0 and ω_1 so

$$
x_0 + x_1 = -\frac{2k(y_0 - kx_0)}{1 + k^2}.
$$

From this relation we get the expression for x_1 . Similarly we proceed for y_1 . simulation, the pedagogic value is not quite the same), but *complementing* it. It can as well be useful from this relation we get the expression for x_1 . Similarly we proceed for y_1 .

Lemma 3. Given an ellipse e_1 : $x^2/a_1^2 + y^2/b_1^2 = 1$ and a point $A_0(\cos\varphi_0, \sin\varphi_0)$ on the unit circle denote by defining b. Given an employer c_1 , $x/a_1 + y/a_1 - 1$ and a point $A_0(\cos \varphi_0, \sin \varphi_0)$ on the same $\mathcal{L}_{\text{true}}$ are the other surface of the lens – again mathematics is required to calculate the angle in angle i

$$
t: y - y_0 = k(x - x_0)
$$

any line from A_0 and let A_1 the second point of intersection of this lines with the circle $k(O, 1)$: $x^2 + y^2 = 1$, such that $A_1(\cos \varphi, \sin \varphi)$. Then the relations of Lemma [2](#page-3-1) take the form we have b_{α} is b_{α} and the set α from the equations of Behmann α and the form we have

$$
\theta=\frac{\varphi+\varphi_0-\pi}{2}+m\pi, m\in\mathbb{Z},
$$

where **2** Extending the surface of θ and θ

$$
\theta = \arctan k.
$$

Proof. We have the relations

$$
\frac{k^2 - 1}{k^2 + 1} = -\cos(2\theta), \quad \frac{2k}{k^2 + 1} = \sin(2\theta).
$$

Making the substitution

$$
x_1 = \cos \varphi, y_1 = \sin \varphi
$$

we find

$$
\cos \varphi = -\cos(2\theta)\cos\varphi_0 - \sin(2\theta)\sin\varphi_0 =
$$

= $\cos(2\theta + \pi)\cos\varphi_0 + \sin(2\theta + \pi)\sin\varphi_0 = \cos(2\theta + \pi - \varphi_0),$
 $\sin \varphi = -\sin(2\theta)\cos\varphi_0 + \cos(2\theta)\sin\varphi_0 =$
= $\sin(2\theta + \pi)\cos\varphi_0 - \cos(2\theta + \pi)\sin\varphi_0 = \sin(2\theta + \pi - \varphi_0),$

and these relations lead simply to the needed relation

$$
2\theta + \pi - \varphi_0 = \varphi + 2m\pi, m \in \mathbb{Z}.
$$

This completes the proof.

 \Box

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Lemma 4. Given an ellipse e_1 : $x^2/a_1^2 + y^2/b_1^2 = 1$ and a point $A_0(\cos\varphi_0, \sin\varphi_0)$ denote by **Modelling optical lenses with Dynamic Geometry Software**

$$
t: y - y_0 = k(x - x_0)
$$

a line through A_0 and by A_1 the point of the intersection of this line with the circle $e : x^2 + y^2 = 1$, such that $A_1(\cos\varphi, \sin\varphi)$. Then t is tangent to e_1 if and only if $\lim_{\omega \to 0} \frac{1}{\omega}$ (core φ), $\lim_{\omega \to 0} \frac{1}{\omega}$ is experiment and ω of $\frac{1}{\omega}$ need a lot of order and $\frac{1}{\omega}$

we have we have ℓ

$$
\cos^2\left(\frac{\varphi-\varphi_0}{2}\right) = b_1^2 \sin^2\left(\frac{\varphi+\varphi_0}{2}\right) + a_1^2 \cos^2\left(\frac{\varphi+\varphi_0}{2}\right) = (a_1^2 - b_1^2) \cos^2\left(\frac{\varphi+\varphi_0}{2}\right) + b_1^2.
$$

Proof. From Lemma [1](#page-3-2) we see that we need to transform $(y_0 - kx_0)^2$ into a function of φ and φ_0 . Indeed, we have α indeed we have α μ_0 . through a lens with the help of dynamic geometry software (μ_0).

$$
y_0 - kx_0 = \frac{\cos\theta \sin\varphi_0 - \sin\theta \cos\varphi_0}{\cos\theta} = \frac{\sin(\varphi_0 - \theta)}{\cos\theta}.
$$
 (11)

Using now the relation

Using now the relation
$$
\theta = \frac{\varphi + \varphi_0 - \pi}{2} + m\pi, m \in \mathbb{Z},
$$

from Lemma [1,](#page-3-2) we see the the numerator in (11) is

$$
\sin(\varphi_0 - \theta) = \sin\left(\frac{\varphi_0 - \varphi + \pi}{2} - m\pi\right) = (-1)^m \cos\left(\frac{\varphi_0 - \varphi}{2}\right)
$$

while the denominator becomes

$$
\cos \theta = \cos \left(\frac{\varphi + \varphi_0 - \pi}{2} + m\pi \right) = (-1)^m \sin \left(\frac{\varphi + \varphi_0}{2} \right)
$$

so we find **2.1 Reflection**

$$
\sin^2\left(\frac{\varphi+\varphi_0}{2}\right)(y_0-kx_0)^2=\cos^2\left(\frac{\varphi-\varphi_0}{2}\right).
$$

Applying Lemma [1](#page-3-2) combined with the above relations, we complete the proof of the Lemma.

Remark 1. We can rewrite the relations of Lemma $\frac{1}{4}$ in different ways using the formula

$$
\cos^2 \alpha = \frac{1 + \cos(2\alpha)}{2},
$$

also as

$$
\cos(\varphi - \varphi_0) = c^2 \cos(\varphi + \varphi_0) + D,\tag{12}
$$

 \Box

or

$$
(1 - c2) cos \varphi cos \varphi_0 + (1 + c2) sin \varphi sin \varphi_0 = D,
$$
\n(13)

where

$$
c^2 = a_1^2 - b_1^2, D = a_1^2 + b_1^2 - 1.
$$
\n(14)

Figure 4: The meaning of the assumption $\Delta A_0 B_0 C_0$ is circumscribed on e_1 ? Figure 4: The meaning of the assumption $\Delta A_0 B_0 C_0$ is circumscribed on e_1 ?

3 Proof of Poncelet theorem using trigonometric functions which the light is reflected and reflecting and reflection calculation calculation calculation calculating the

We take a point $A_0(cos\varphi_0, sin \varphi_0)$ on the unit circle and find of two tangent lines t_1, t_2 through A_0 to the ellipse

$$
e_1 = \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1.
$$

Then we find the intersection points of t_1, t_2 with the unit circle (see Figure [4\)](#page-6-0) and denote the two intersection points (different from A_0) by

$$
B_0(cos\varphi_1, sin \varphi_1), C_0(cos\varphi_2, sin \varphi_2).
$$

First, let us express the assumption of Poncelet's theorem that there exists at least one triangle $\Delta A_0 B_0 C_0$ inscribed in the unit circle, i.e. When a ray of light hits a plane glass surface, a part of it is reflected. The *law of reflection* says that first, let us express the assumption of Poncelet's theorem that there exists at least one

$$
A_0(\cos\varphi_0, \sin\varphi_0), B_0(\cos\varphi_1, \sin\varphi_1), C_0(\cos\varphi_2, \sin\varphi_2), 0 \le \varphi_0 < \varphi_1 < \varphi_2 \le 2\pi
$$

and circumscribed on the inner ellipse e_1 Since A_0B_0 is tangent to e_1 we know that:

$$
\cos^2\left(\frac{\varphi_1 - \varphi_0}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\varphi_1 + \varphi_0}{2}\right) + b_1^2\tag{15}
$$

(this is due to Lemma [4\)](#page-4-0). Similarly, the fact that A_0C_0 and B_0C_0 are tangent to e_1 , and Lemma [4](#page-4-0) imply

$$
\cos^2\left(\frac{\varphi_2 - \varphi_0}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\varphi_2 + \varphi_0}{2}\right) + b_1^2.
$$
 (16)

$$
\cos^2\left(\frac{\varphi_2 - \varphi_1}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\varphi_2 + \varphi_1}{2}\right) + b_1^2.
$$
 (17)

We can unify all these relations into one

$$
\cos^2\left(\frac{\varphi_j - \varphi_\ell}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\varphi_j + \varphi_\ell}{2}\right) + b_1^2, \quad 0 \le j \neq \ell \le 2. \tag{18}
$$

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What we know from the assumptions the Poncelet theorem and what we have to prove? **Modelling optical lenses with Dynamic Geometry Software**

Take any point $A(cos\psi_0, sin \psi_0)$ on the unit circle and find of two tangent lines t_1, t_2 through A_0 to the ellipse

$$
e_1: \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1.
$$

Then we find the intersection points of t_1, t_2 with the unit circle (see Figure [5\)](#page-7-0) and denote the two intersection points (different from \overrightarrow{A}) by $\frac{1}{\sqrt{2}}$

 $B(cos\psi_1, sin \psi_1), C(cos\psi_2, sin \psi_2).$ $\mathcal{L}(\cos\varphi_1, \sin\varphi_1), \cos\varphi_2, \sin\varphi_2.$

Figure 5: Two sides tangent \Rightarrow the third side is also tangent. When a ray of light hits a plane glass surface, a part of it is reflected. The *law of reflection* says that

Since AB is tangent to e_1 we know that:

$$
\cos^2\left(\frac{\psi_1 - \psi_0}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\psi_1 + \psi_0}{2}\right) + b_1^2\tag{19}
$$

(this is due to Lemma [4\)](#page-4-0). Similarly, the fact that A_0C_0 and B_0C_0 are tangent to e_1 , and Lemma [4](#page-4-0) imply

$$
\cos^2\left(\frac{\psi_2 - \psi_0}{2}\right) + (a_1^2 - b_1^2)\cos^2\left(\frac{\psi_2 + \psi_0}{2}\right) = b_1^2.
$$
 (20)

So we summarize all assumptions of Poncelet's theorem and can say that [\(18\)](#page-6-1), [\(19\)](#page-7-1) and [\(20\)](#page-7-2) are satisfied.

What we have to prove?

Having in ming again Lemma [4](#page-4-0) we see that our purpose is to show that

$$
\cos^2\left(\frac{\psi_2 - \psi_1}{2}\right) = (a_1^2 - b_1^2)\cos^2\left(\frac{\psi_2 + \psi_1}{2}\right) + b_1^2.
$$
 (21)

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This relation can be rewritten as

$$
(1 - c2) cos ψ_2 cos ψ_1 = (1 + c²) sin ψ_2 sin ψ_1 + D, (22)
$$

where

where

$$
c^2 = a_1^2 - b_1^2, D = a_1^2 + b_1^2 - 1.
$$
 (23)

according to Remark [1.](#page-5-1) $\frac{1}{\sqrt{2}}$ making to show that $\frac{1}{\sqrt{2}}$

Now we are in position to apply the trigonometric lemma from the appendix and conclude that

$$
\cos^2\left(\frac{\psi_2 - \psi_1}{2}\right) = \frac{4c^2D^2}{(1 - c^2)^2(1 + c^2)^2}\cos^2\left(\frac{\psi_2 + \psi_1}{2}\right) + \frac{D^2}{(1 + c^2)^2}.
$$
 (24)

Comparing this relation with (21) we see that the following conditions

$$
4D^2 = (1 - c^2)^2 (1 + c^2)^2, \ D^2 = b_1^2 (1 + c^2)^2 \tag{25}
$$

are required. This relations and (23) lead to the following sufficient condition light hits the glass surface of an optical lens, a part of it gets reflected back in a certain angle, and

$$
a_1 + b_1 = 1 \tag{26}
$$

that implies \triangle ABC is circumscribed on e_1 . The condition [\(23\)](#page-8-0) is also necessary for the fulfillment $\frac{1}{\sqrt{2}}$ of the property near the lens of the lens of the light being more off-centre, the calculations and light being more of Γ

• there exists a triangle $\Delta A_0 B_0 C_0$ circumscribed on e_1 . \bullet there exists a triangle $\Delta A_0 D_0 C_0$ circumscribed on e_1 .

If there exists at least one $\Delta A_0 B_0 C_0$ circumscribed on e_1 , then [\(26\)](#page-8-1) and hence ΔABC is circumscribed on e_1 .

This completes the proof of the Theorem.

4 Appendix: Trigonometric Lemma a ray of light his reflection says it is reflected. The analysis of reflection

Lemma 5. Suppose the angle of incidence (between the ray of light and the *normal*) is equal to the angle of reflection:

$$
\sin\left(\frac{\psi_1 - \psi_2}{2}\right) \neq 0, \cos\left(\frac{\psi_1 + \psi_2}{2}\right) \neq 0, \cos\psi_0
$$

and

$$
\begin{cases}\n(1 - c^2)\cos\psi_1\cos\psi_0 + (1 + c^2)\sin\psi_1\sin\psi_0 = D \n(1 - c^2)\cos\psi_2\cos\psi_0 + (1 + c^2)\sin\psi_2\sin\psi_0 = D\n\end{cases}
$$
\n(27)

Then

$$
(1 - c2) tan \left(\frac{\psi_1 + \psi_2}{2}\right) = (1 + c2) tan \psi_0
$$
 (28)

and moreover

$$
\cos^2\left(\frac{\psi_2 - \psi_1}{2}\right) = \frac{4c^2D^2}{(1 - c^2)^2(1 + c^2)^2}\cos^2\left(\frac{\psi_2 + \psi_1}{2}\right) + \frac{D^2}{(1 + c^2)^2}.
$$
 (29)

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Proof. Take the difference between the relations in [\(27\)](#page-8-2). We get **Modelling optical lenses with Dynamic Geometry Software** Andreas Ulovec

$$
-(1-c^2)\sin\left(\frac{\psi_1-\psi_2}{2}\right)\sin\left(\frac{\psi_1+\psi_2}{2}\right)\cos\psi_0+(1+c^2)\sin\left(\frac{\psi_1-\psi_2}{2}\right)\cos\left(\frac{\psi_1+\psi_2}{2}\right)\sin\psi_0=0.
$$

The assumption

$$
\sin\left(\frac{\psi_1 - \psi_2}{2}\right) \neq 0
$$

implies that $\sum_{i=1}^{\infty}$

$$
(1 - c2) sin \left(\frac{\psi_1 + \psi_2}{2}\right) cos \psi_0 = (1 + c2) cos \left(\frac{\psi_1 + \psi_2}{2}\right) sin \psi_0.
$$

This proves [\(28\)](#page-8-3). The other relation can be obtained following the plan

- first equation in $(27) \times \sin \psi_2$ second equation in $(27) \times \sin \psi_1$; simulation, the pedagogic value is not quite the same), but *complementing* it. It can as well be useful
- first equation in $(27) \times \cos \psi_2$ second equation in $(27) \times \cos \psi_1$. light hits the glass surface of an optical lens, a part of it gets reflected back in a certain angle, and

In this way we get light reaches the other surface of the lens – again mathematics is required to calculate the angle in

$$
2D\sin\left(\frac{\psi_2-\psi_1}{2}\right)\cos\left(\frac{\psi_2+\psi_1}{2}\right) = 2(1-c^2)\sin\left(\frac{\psi_2-\psi_1}{2}\right)\cos\left(\frac{\psi_2-\psi_1}{2}\right)\cos\psi_0,
$$

$$
-2D\sin\left(\frac{\psi_2-\psi_1}{2}\right)\sin\left(\frac{\psi_2+\psi_1}{2}\right) = -2(1+c^2)\sin\left(\frac{\psi_2-\psi_1}{2}\right)\cos\left(\frac{\psi_2-\psi_1}{2}\right)\sin\psi_0,
$$

so using the assumption

$$
\sin\left(\frac{\psi_1 - \psi_2}{2}\right) \neq 0
$$

we find **2.1 Reflection**

$$
\frac{D}{1-c^2}\cos\left(\frac{\psi_2+\psi_1}{2}\right) = \cos\left(\frac{\psi_2-\psi_1}{2}\right)\cos\psi_0,
$$

$$
\frac{D}{1+c^2}\sin\left(\frac{\psi_2+\psi_1}{2}\right) = \cos\left(\frac{\psi_2-\psi_1}{2}\right)\sin\psi_0.
$$

Taking the sum of squares of these identities we obtain

$$
\frac{D^2}{(1-c^2)^2} \cos^2\left(\frac{\psi_2 + \psi_1}{2}\right) + \frac{D^2}{(1+c^2)^2} \sin^2\left(\frac{\psi_2 + \psi_1}{2}\right) = \cos^2\left(\frac{\psi_2 - \psi_1}{2}\right)
$$

and this equation yields [\(29\)](#page-8-4).

This completes the proof of the Lemma.

 \Box

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