Lifelong Learning Programme

DynaMAT

Euclidean Eggs

Freyja Hreinsdóttir

University of Iceland

1 Introduction

Using geometric computer programs such as GeoGebra it is very easy to do constructions with a ruler and a compass. In particular it is easy to draw arcs and circles and putting those together in certain ways one can create objects with different shapes such as eggs.

2 Arcs and circles

To define a circle we need two parameters, the centre point and the radius. What do we need to define an arc? An arc is basically a part of a circle, so we need the same information to know *which* circle the arc is a part of. But we also need information on the *length* and *position* of the arc on the circle, i.e. given the circle we need to know where the arc starts and where it ends.

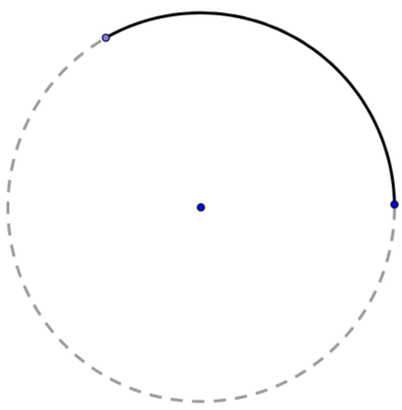

Fig. 1 Arc on a circle

In GeoGebra there are several tools to create circles and arcs:

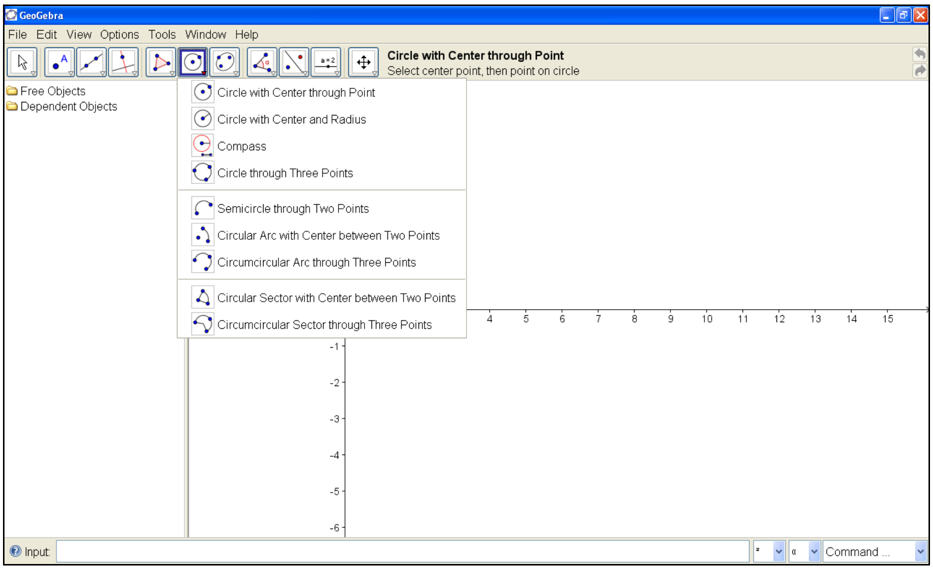

Fig.2 Screenshot showing tools in GeoGebra for constructing arcs and circles

Task: Create several arcs and circles in GeoGebra. Make some of the arcs join each other and use different colours and styles (fillings) to make a pretty picture. To change the colours and styles you right click on an object and select “object properties” in the menu that opens.

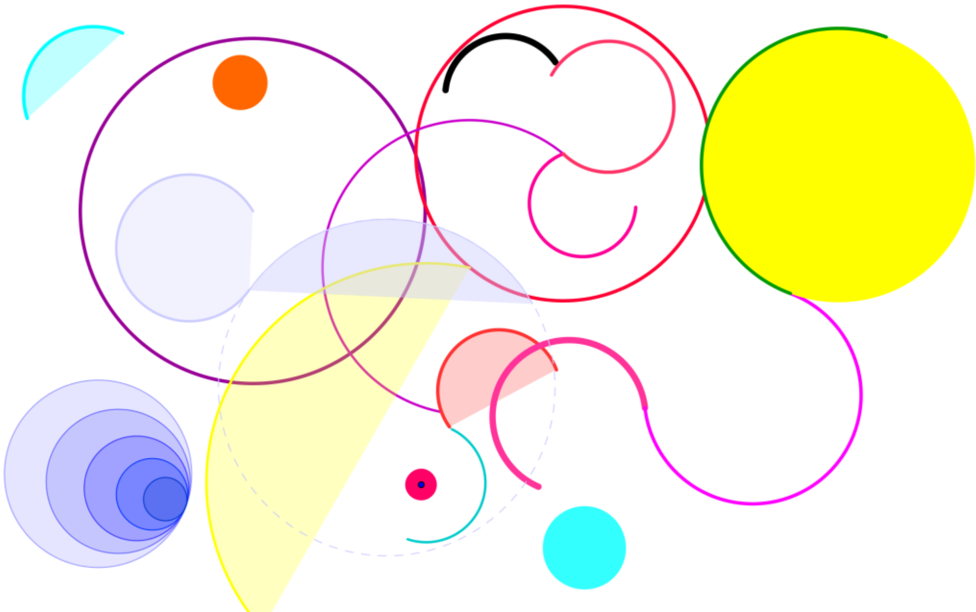

Fig. 3 Picture made from arcs and circles

Notice that some of the arcs in the picture are joined in a *smooth* way, that is there is no bend or break where they meet. We now investigate how this is done.

Task: Open GeoGebra and create two arcs *c* and *d* that meet in one point. Try to move the points defining the arcs in such a way that the meeting is smooth. This task is easier if you draw a line through the meeting point and the center point defining one of the arcs.

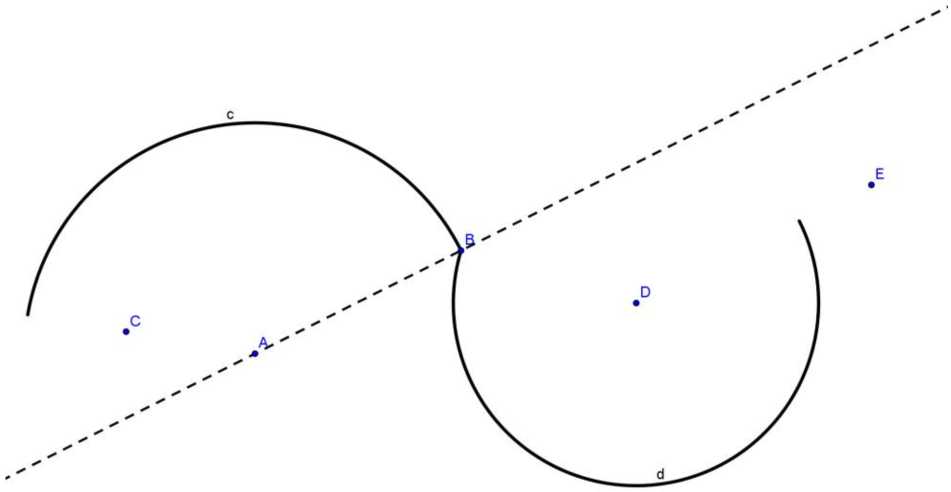

Fig. 4 Two arcs that meet in one point

The point *D* can be moved such that the arcs meet in a smooth way resulting in the picture below.

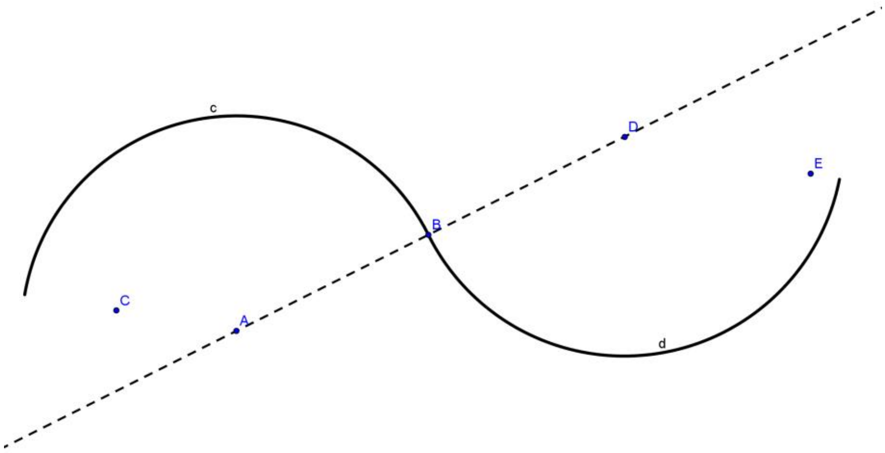

Fig. 5 Two arcs meeting in a smooth way

You will probably notice that a necessary condition for smoothness is that the **two center points (of the circles defining the arcs) and the meeting point lie on a line**. This condition is also sufficient if in your picture the two arcs lie on the opposite sides of the line.

Using this principle we can create pictures like:

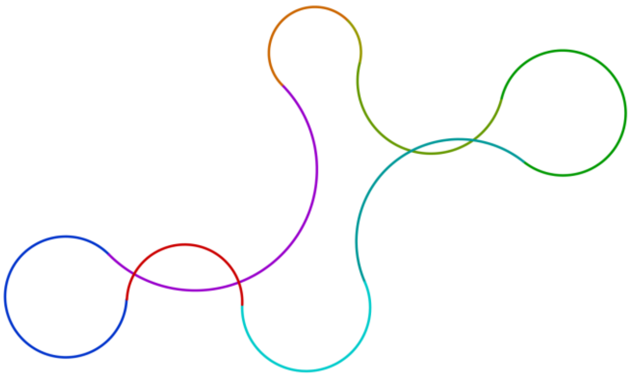

Fig. 6 Many arcs meeting in a smooth way

This is done with the help of many lines and circles that are hidden in the final picture but shown in the picture below. Because of the dynamic properties of GeoGebra it is possible to move the points around to get different smooth pictures as long as the requirements are satisfied (the three points are on the same line and the arcs are on the opposite sides of that line).

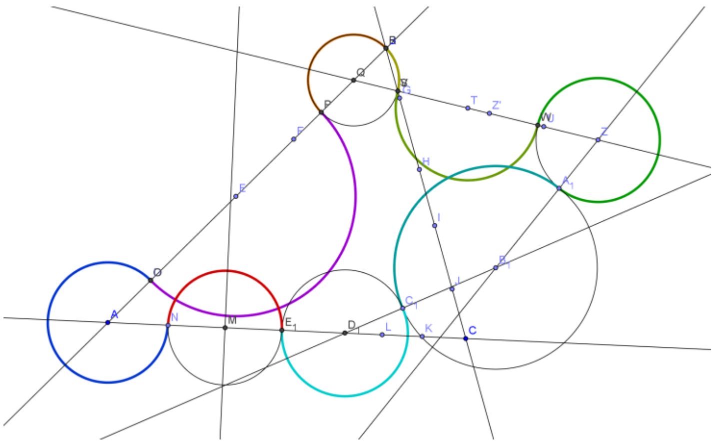

Fig. 7 Many arcs meeting in a smooth way with the help-lines shown

Task: Create your own picture similar to the one above.

4 Eggs

Below you see a picture of several bird eggs. As you can see they are very different in size but similar in shape although some of them are more pointed than the others.

Fig. 8 Eggs (from http://en.wikipedia.org/wiki/Egg_(biology))

We can use arcs to construct egg-shaped objects like the ones in the picture below. How to do this will be explained in the next section.

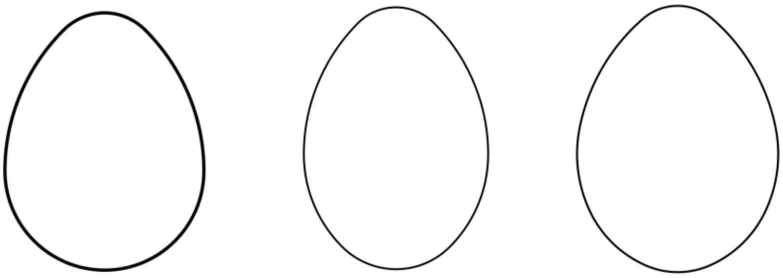

Fig. 9 Moss egg, Four-point egg and Five-point egg created using GeoGebra

5 Euclidean Eggs

In the very nice book *Mathographics* by Robert Dixon [1] there is a section on egg-shaped ovals constructed using a ruler and a compass. The author calls them *Euclidean eggs* and gives pictures of several such eggs although the details of the construction are not given (Dixon (1987), pp. 3 – 11).

Using arcs and circles we can construct these egg shaped curves that resemble cross sections of real eggs. The complexity of these constructions varies greatly. They are relatively easy once the principle has been figured out but the constructions are done in many steps and are quite time consuming.

5.1 Moss Egg

Task: Try to use the picture below to construct your own Moss egg:

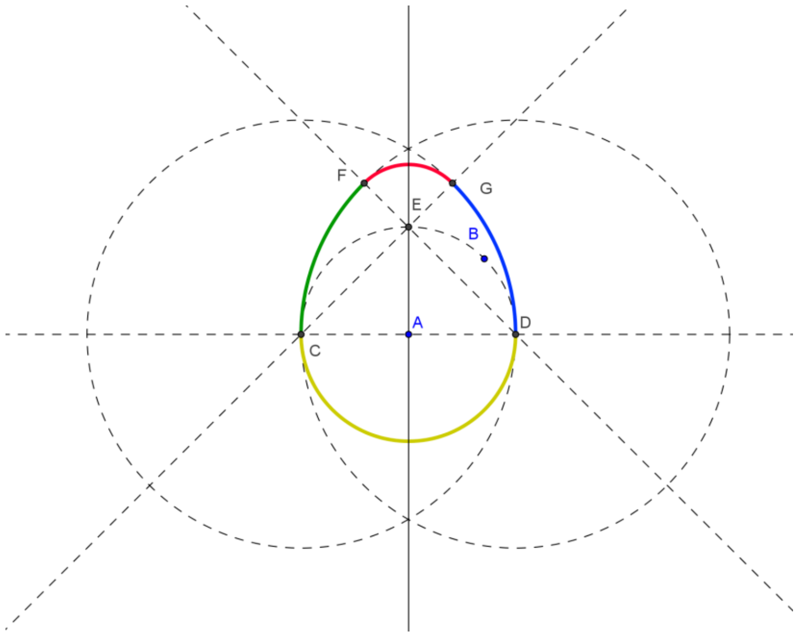

Fig. 10 Moss egg constructed in GeoGebra

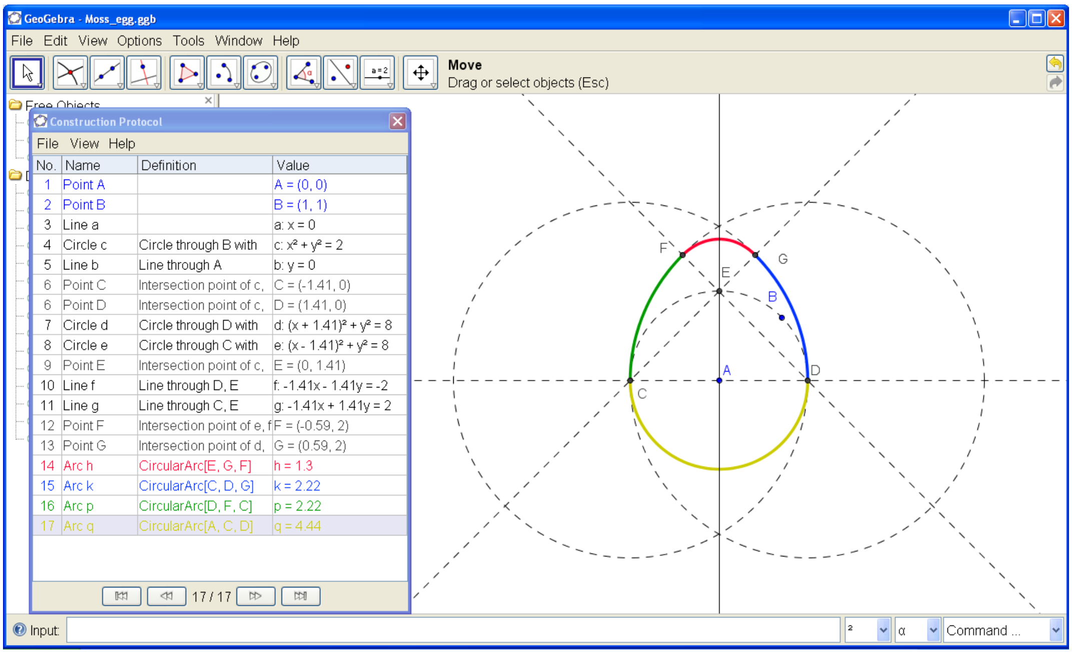

Fig. 11 Screenshot from GeoGebra with the construction protocol

In the construction above the radius of the two large circles is equal to the diameter of the first circle drawn. If we create new points *H* and *I* on the diameter line and go through the construction we get an egg that looks slightly different.

Task: Make a GeoGebra worksheet such that you can vary the location of the center points mentioned above and thus experiment with different shapes of a similar type of egg (hint: use a slider).

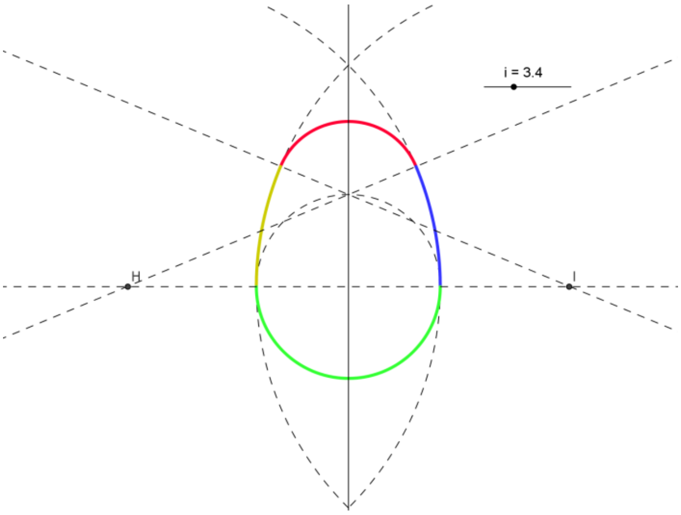

Fig.12 Variations of a Moss egg

5.2 Four-point egg

We are now going to draw the so called *Four-point egg*.

Icons	Construction	Name
	Draw a line. It is easiest to draw a vertical line e.g. the y – axis although coordinates are not needed for this construction.	a
	Define a point on a	A
	Draw a line through A perpendicular to a	b
	Define another point on a	B
	Draw a circle through B with center A	c
	Mark the (other) intersection point of a and c	C
	Draw a circle through B with center C	d
	Mark the intersection points of c and b	D, E
	Draw a line through D and C	e
	Draw a line through E and C	f
	Mark the intersection point of d and e	F
	Mark the intersection point of d and f	G
	Draw a circle through F with center D	g
	Draw a circle through G with center E	h
	Mark the intersection points of g and the line b	H, I
	Mark the intersection points of h and b	J, K

	Draw an arc with center C through G and F	k
	Draw an arc with center D through F and I	p

Now your construction should be approximately as shown in the picture below.

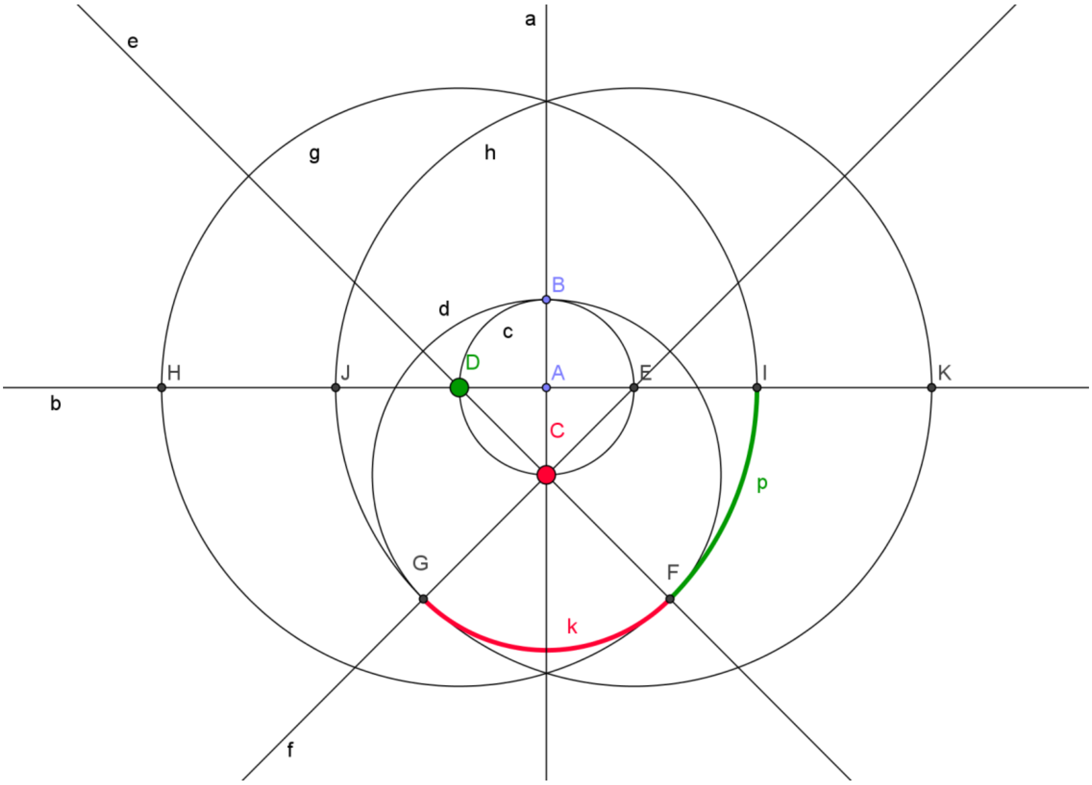

Fig.13 The points *C* and *D* are two of the four points defining the egg. The green and the red arc are parts of the egg corresponding to the two points.

Four-point egg continued

	Construction	Name
	Draw a circle with center A through I and J	q
	Mark the intersection point of a and q	L
	Draw a line through I and L	i
	Draw a line through J and L	j
	Draw a circle with center J through I	r
	Draw a circle with center I through J	s
	Draw a circle with center L through B	t
	Mark the intersection point of the circle r and the line i	N
	Mark the intersection point of the circle s and the line j	P
	Draw the arc with center J through I and N	c_1
	Draw the arc with center L through N and P	d_1

We now have almost completed the construction of the four-point egg. The four points defining the egg are *C*, *D*, *J* and *L* as can be seen in the picture below where color has been used to show the correspondence between the arcs and the points.

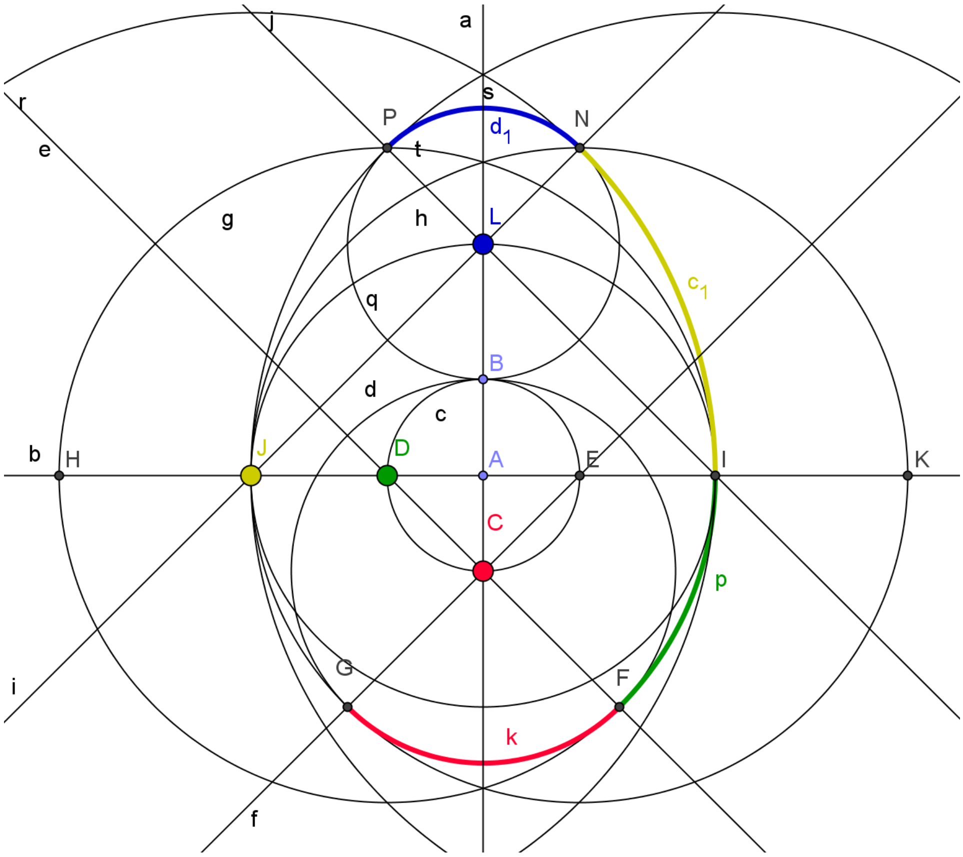

Fig. 14 The four points and corresponding arcs.

The remaining arcs of the egg are drawn by symmetry and if we hide all labels, circles and most lines (replacing some of them with segments) we get the final egg as shown below.

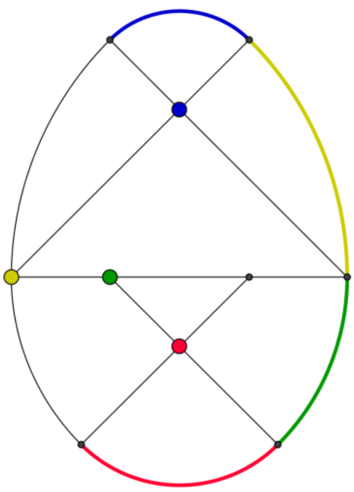

Fig. 15 The four point egg

Note: it is not really necessary to draw all the circles used in the construction above since in some cases we can draw the arc directly.

5.3 The Five-point egg

Below we have a picture of a five-point egg with all help lines and circles needed to draw the egg

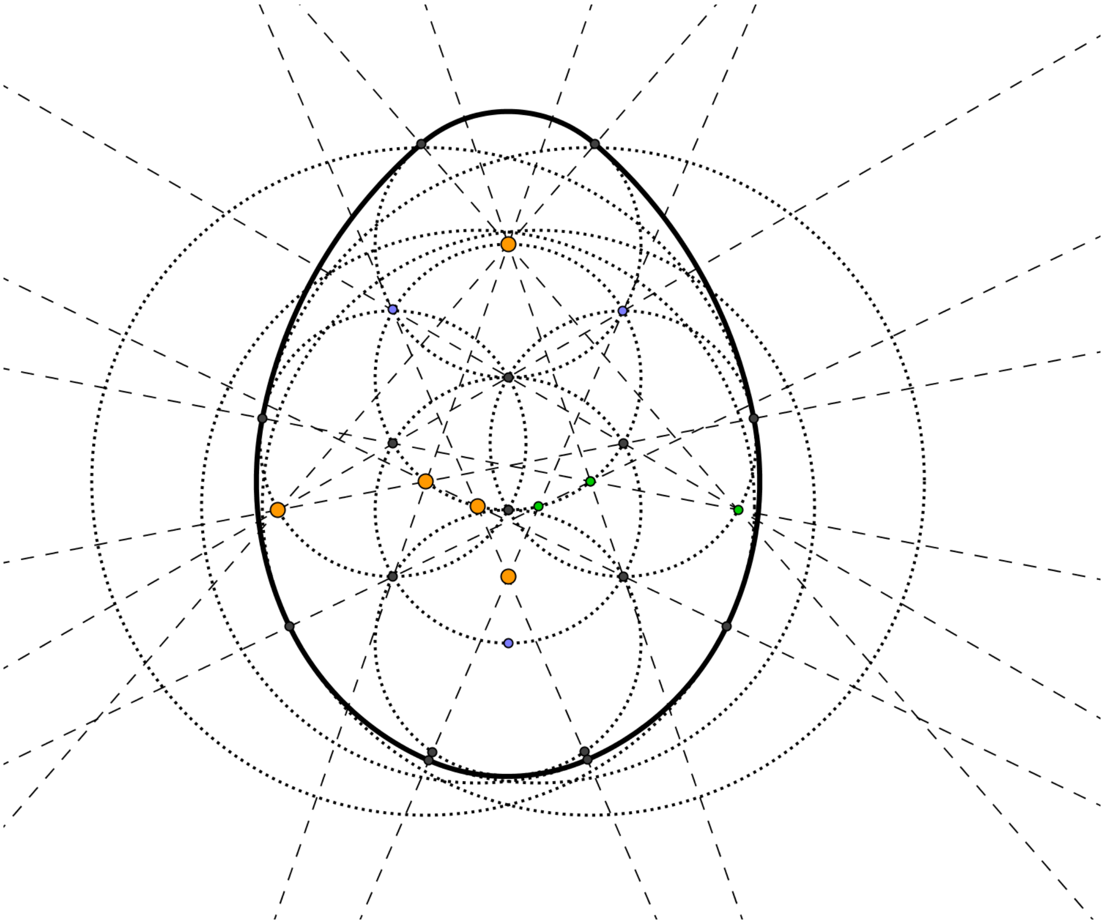

Fig. 16 Five-point egg

The five orange points are the five points defining the arcs on the right side of the egg and the green ones are the points needed to draw the left side. If we hide all circles and lines, replacing some of them with segments we get the picture below showing the five points needed for the right side of the egg.

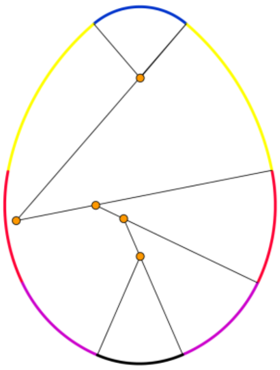

Fig. 17 The five-point egg

Task: Use GeoGebra or some other dynamic geometry software to create a five-point egg. Try dragging the points to see how the shape of the egg changes. Note: if the egg is drawn correctly all the meeting point of different arcs should remain smooth after the dragging.

Experimenting

The bottom half of the five-point egg demonstrates a construction that can easily be continued, i.e. we can pick more points on the line segments and create a spiral-looking figure like:

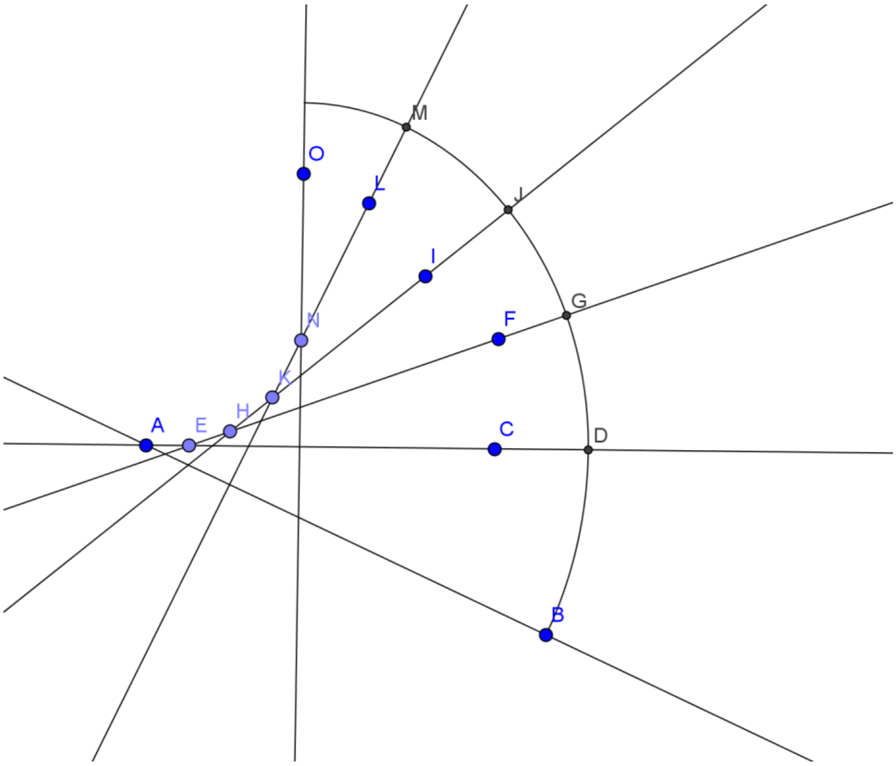

Fig.18 A spiral

This was done using only the line tool, the arc tool and the two point tools. We can drag all the blue points to change the construction. Drawing segments and hiding the lines and points we get the picture below:

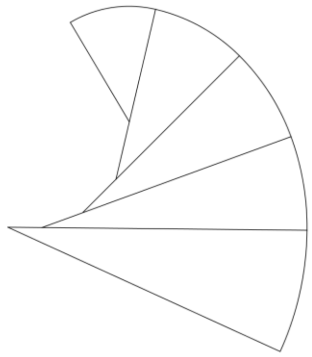

Fig. 19 Spiral above with lines and labels hidden

Task: Create the spiral above. Experiment with colors to make a nicer picture.

Task: At http://mathworld.wolfram.com/ThomsEggs.html there are pictures of Thom's eggs [3]. Construct the two eggs using circles, arcs and lines.

Task: Search for "Golden Egg" on the internet to find a picture of the Golden egg and construct it using circles, arcs and lines (a picture can also be found in Dixon's book [1]).

Task: Experiment with similar constructions and make your own.

References

[1] Dixon, R. *Mathographics*. Basic Blackwell Limited, Oxford, England, 1987.

[2] GeoGebra, downloadable from http://www.geogebra.org.

[3] Weisstein, Eric W. "Thom's Eggs." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/ThomsEggs.html