Investigating 2 by 2 matrices – part I

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1 Introduction

In GeoGebra 4.0 there is a possibility of having two graphic windows instead of only one. These two windows communicate with each other so there is a possibility to investigate the effects of transformations from the plane to the plane. In this chapter we show several possibilities of doing this and in *Investigating 2 by 2 matrices – part II* we continue our study.

2 Theory

To study a function f from R to R we often draw its graph either by hand or by using a computer program. The graph is then the set of all points of the form $(x, f(x))$. If we have a function $T: \mathbb{R}^2 \to \mathbb{R}^2$ we would need 4 dimensions to study a similar point set i.e. the set $(x, y, T_1(x,y), T_2(x,y))$, where T_1 gives the first coordinate of the image and T_2 the second, so this is not possible. However, if we imagine that we have 2 copies of the plane we can study images of objects in one of the planes in the other plane.

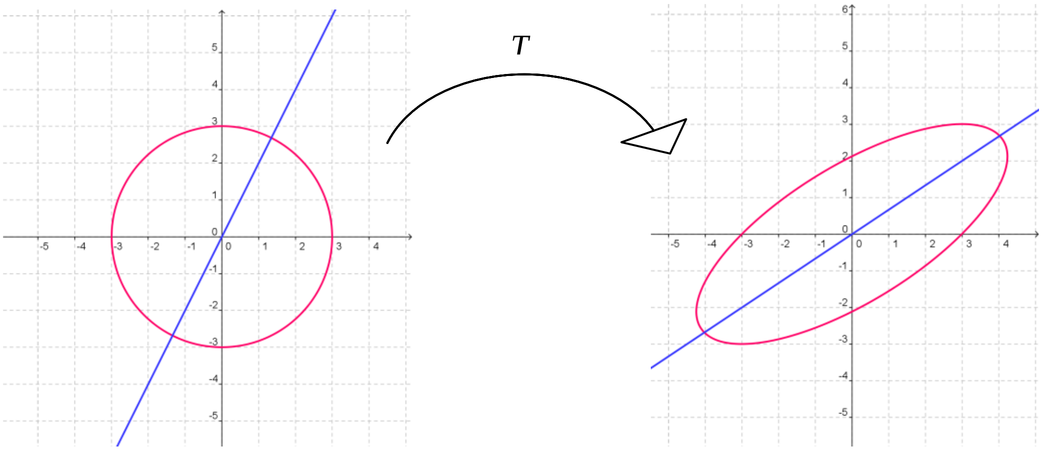


Fig. 1 A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

A function $T: \mathbb{R}^2 \to \mathbb{R}^2$ is usually called a *transformation* from $\mathbb{R}^2$ to $\mathbb{R}^2$. Of particular interest are transformations that have the following two properties for all vectors *u* and *v* in $\mathbb{R}^2$ and all scalars (real numbers) *c*:

a) *T(cu) = c T(u)* - additivity

b) *T(u + v) = T(u) + T(v)* - homogeneity

Such a transformation is called a *linear transformation* and it has the property that it can be described by a matrix i.e. there exists a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ such that $T(u) = Au$. Conversely we have that, any matrix defines a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ so the study of matrices and the study of linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ are equivalent (for more details on this see any linear algebra book e.g. [1]).

Examples of linear transformations include contractions/dilations, compressions, reflections, rotations and shears. These will be studied in more detail in the chapter *Investigating 2 by 2 matrices – part II*.

3 Using GeoGebra

Below we have a screenshot from GeoGebra:

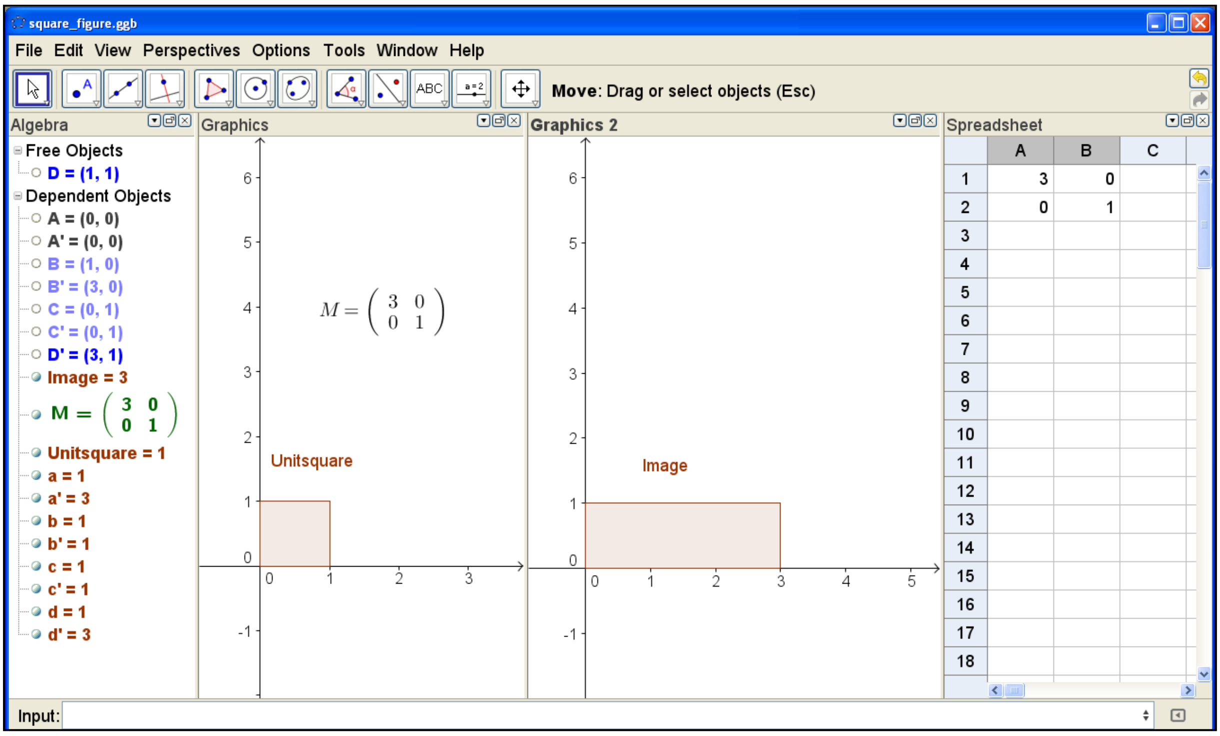


Fig. 2 The image of the unit square under the linear transformation defined by the matrix *M*.

The second graphics view and the spreadsheet are made visible under *view*. Here we have defined the unitsquare using the polygon tool and renamed it (from *poly1*) to have the name *Unitsquare*. The matrix *M* was defined using the spreadsheet. After the matrix has been defined the command *ApplyMatrix* is used to get the image of the unit square under the transformation defined by the matrix *M*.

To define the matrix we type its elements into the spreadsheet, use the mouse to highlight them, right click and choose *Create*. Then a new list opens where we choose *Matrix*.

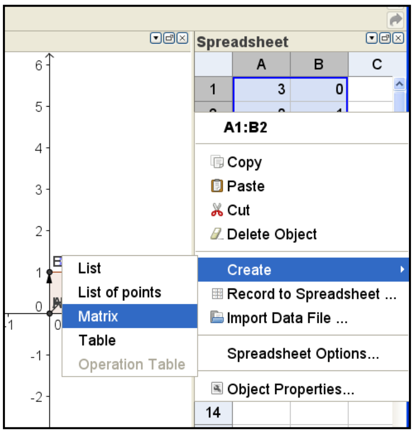


Fig. 3 Creating a matrix using the spreadsheet.

Another, perhaps easier, way to define a matrix is to type it in directly into the input field as a list of lists, that is type $\{\{3,0\},\{0,1\}\}$.

Task: Create the worksheet above. Experiment by changing the definition of the matrix in the spreadsheet.

Note: after you choose *Graphics 2* under *View* the second view opens in front of the first one.

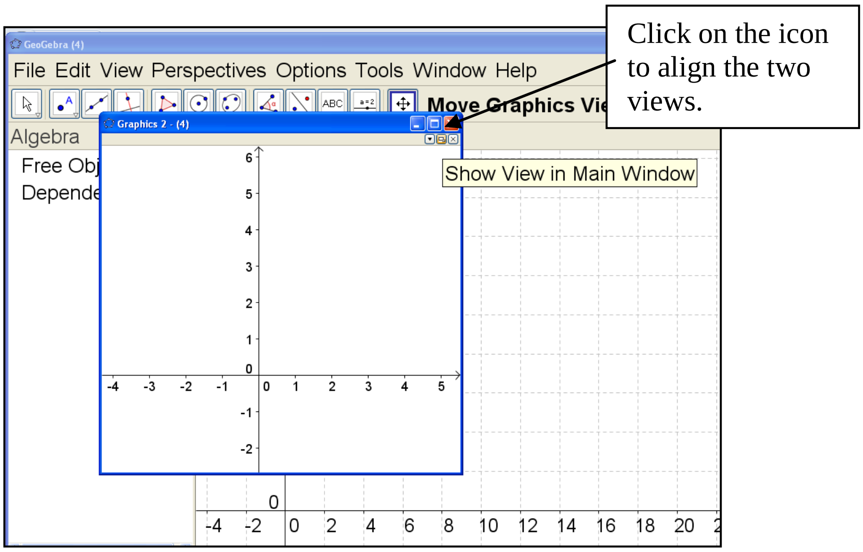


Fig. 4 The second Graphics view.

A graphics view is made active by clicking on it so if you click on *Graphics view 1* and then give a command in the input field the result of the command will appear in *Graphics view 1*. If something ends up in the wrong view, this can be changed under the *advanced* tab in properties (accessible via a *right click* on any object) so we can easily make corrections afterwards.

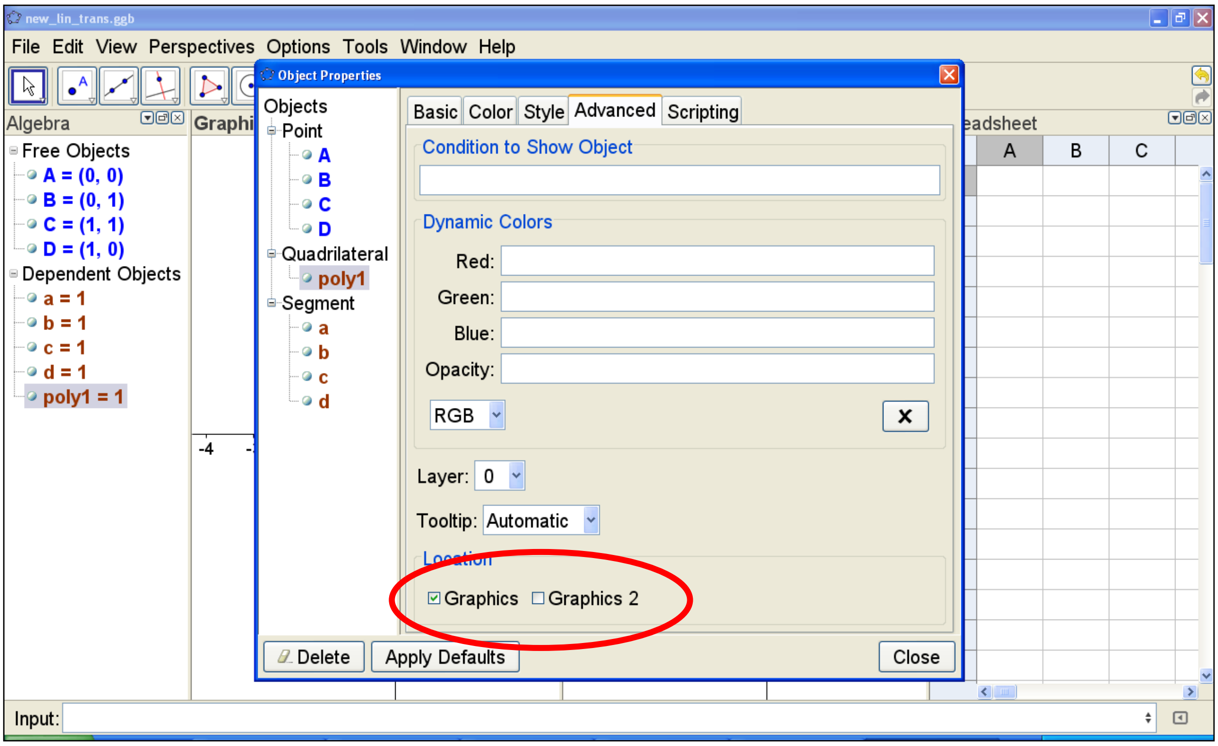


Fig. 5 How to move objects between graphic views.

4 Additivity and homogeneity of linear transformations

For any linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ we have that the following properties hold:

a) *T(cu) = c T(u)* – additivity

b) *T(u + v) = T(u) + T(v)* – homogeneity

for all vectors *u* and *v* in $\mathbb{R}^2$ and all scalars (real numbers) *c*.

These properties are easily demonstrated using the two graphics windows in GeoGebra:

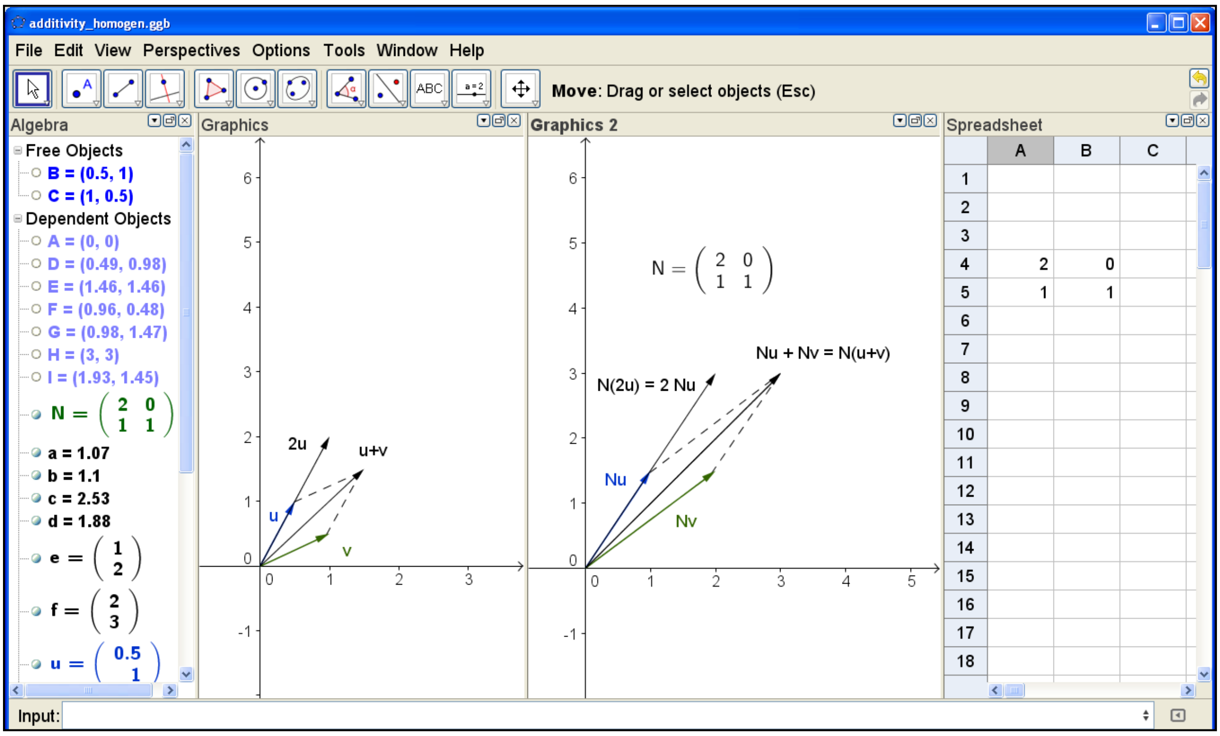


Fig. 6 Demonstrating additivity and homogeneity

5 Linear Transformations and lines

We now investigate the effect a linear transformation has on a line. Since we want to vary our matrix we define it using sliders, *a, b, c* and *d*. This is done by selecting the tool [slider tool] and clicking on the graphics view. Once we have the sliders we define the matrix in the input field by $\{\{a,b\},\{c,d\}\}$. Then a matrix called *matrix1* is created. By right clicking on *matrix1* in the *algebra view* we can rename it to say *M*. We open *Graphics 2* by selecting it under *View*.

Now we define a line *e* through the points (0,0) and (1,1) using the line tool [line tool]. Put the values of the sliders $a = 1$, $b = 0$, $c = 0$, $d = 1$, right click on the moving tool [move tool] and then on *Graphics 2* to make it active. If we now type in the input field *ApplyMatrix[M, e]* we get the image of the line *e* under the transformation *M* in *Graphics 2*. Since our *M* is the identity matrix we of course get the same line as before.

Task: Open a GeoGebra sheet and go through the construction above. Experiment with the values of *a*, *b*, *c* and *d* and observe the effect on the line. In particular put *a = 1, b =-1, c = 1* and *d = -1*. What happens with the line *Me* in *Graphics 2*?

Define a line *f*, through the points (2,0) and (0,4) and apply the matrix to get a new line in *Graphics 2*. Experiment as before with *a*, *b*, *c* and *d*. For which values of these parameters is the line *Mf* undefined?

Task: Define two lines that are parallel to the line *e* (this can easily be done by using the parallel line tool ) and apply the matrix to these. What do you notice about the images of these lines?

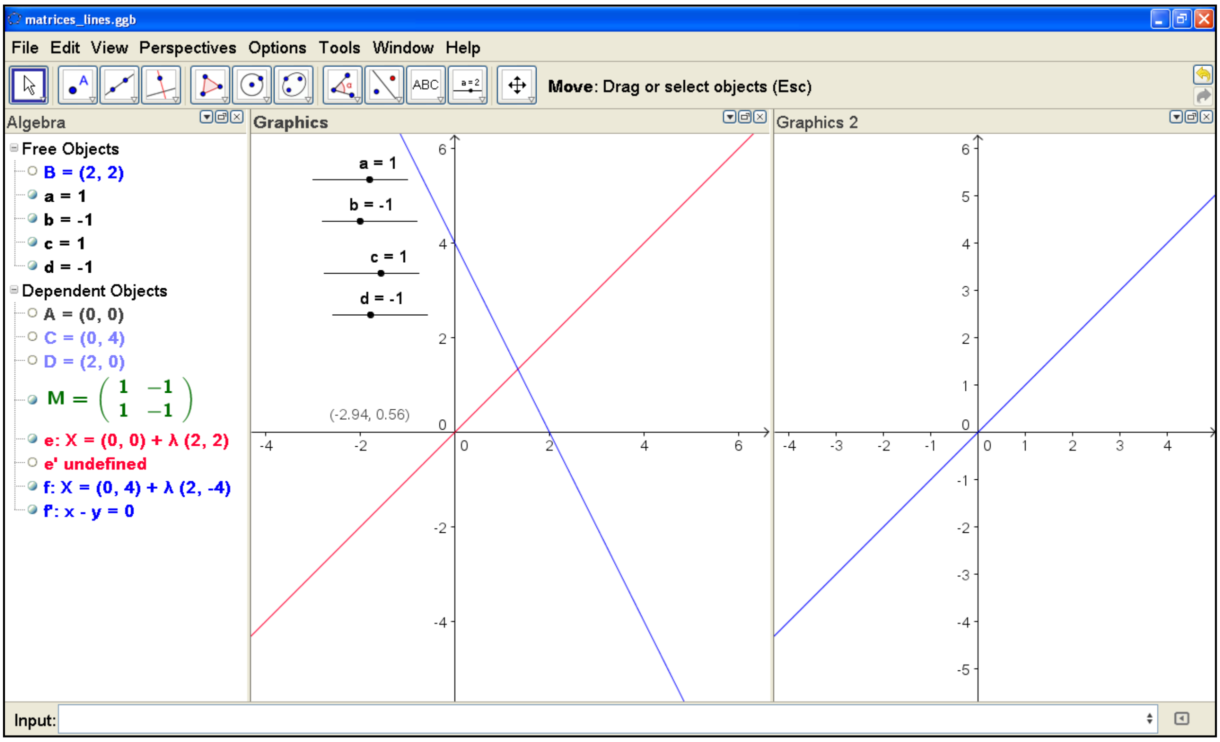


Fig. 7 The equations of the lines in the *Algebra View* are on *parametric form*. This is obtained by right clicking on the equation and choosing this form.

6 Another way to consider transformations

There is another way to see the effect of linear transformations on lines. If we consider for instance the transformation *T* defined by the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we can write it as

$$T_1\,(x, y) = \;ax + by$$

$$T_2(x, y) = \;cx + dy.$$

In our worksheet from before we delete the images from *Graphics 2* (it should now be blank) but keep the lines in *Graphics View 1*. We use the point tool to create a point *E* on the line *e* and click on *Graphics 2* to make it active. Write in the input field *(a*x(E)+b*y(E), c*x(E)+d*y(E))*. This creates a point *F* (in *Graphics 2*) that is the image of *E* under the transformation defined by *M*. Using the moving tool we can now move the point *E* along the line *e* and see its image *F* move in *Graphics 2*. To see the image more clearly we right click on the point *F* and select *Trace on*. If we now move the point *E*, the point *F* traces out a line in *Graphics 2*.

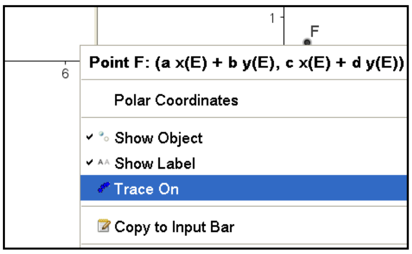


Fig. 8 Right click on any object to select *Trace on*. If the object is moved after that it will leave a trace in the graphics view. To clear the trace we can use  *Refresh Views* under *View*.

Task: Try out the construction above for several different values of *a*, *b*, *c* and *d*.

7 Non-linear transformations

The method above can be used for *any* transformation, even non-linear ones. Consider for instance the transformation U given by

$$(x, y) \to (x \cdot y, y).$$

We can continue with the worksheet above and the point E on the line e. We get $G = U(E)$ by typing in the input field

$$(x(E) * y(E), y(E)).$$

If we put the trace on G and move the point E along the line e then $U(e)$ is traced out in *Graphics 2*.

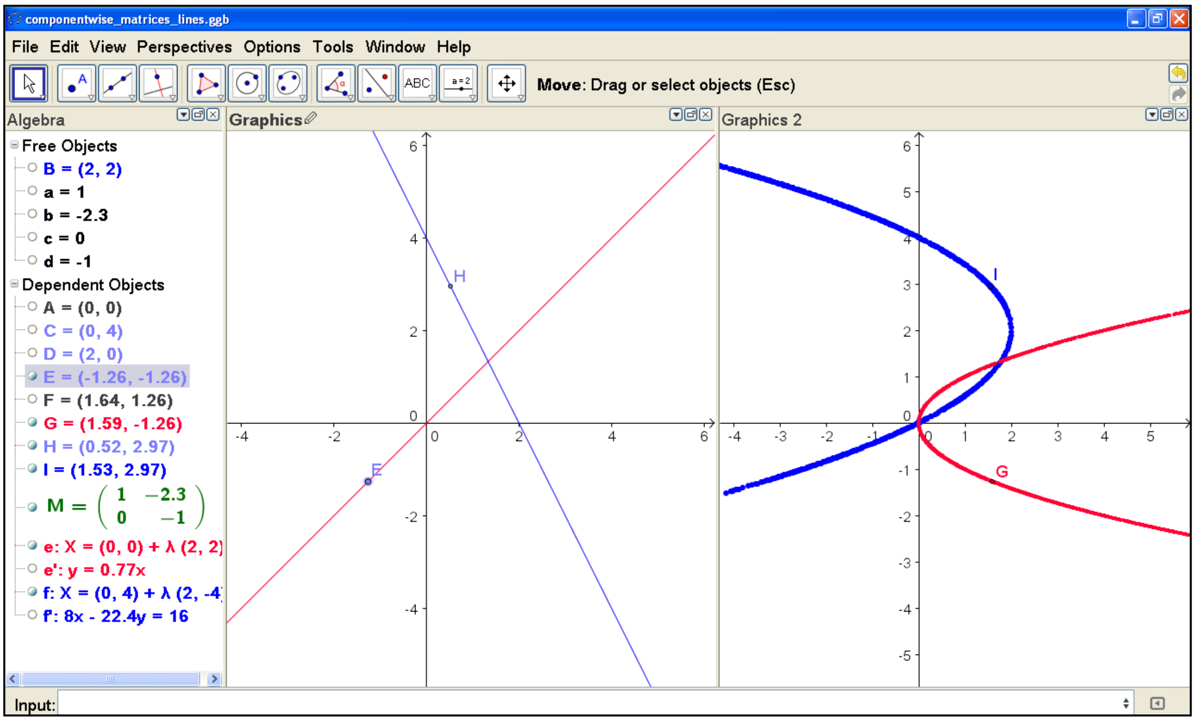


Fig. 10 The image of the lines under the non-linear transformation $(x, y) \to (x \cdot y, y)$.

Not surprisingly the images are parabolas where the x – coordinate is given by a second degree expression in y. The line f has the equation $y = -2x + 4$ or $x = \frac{4-y}{2}$ and $U\left(\left(\frac{4-y}{2}, y\right)\right)$ $= \left(\frac{4-y}{2} \cdot y, y\right)$.

Task: Investigate the images of the lines under the transformation given by $(x, y) \to (x^2, y^2)$. Are the images of the two lines similar? What is the difference and why does it occur?

Invent your own transformations and study the images of the lines under these.

8 Matrices and circles

We now investigate how the image of a point on a circle under a transformation given by a diagonal matrix traces out an ellipse.

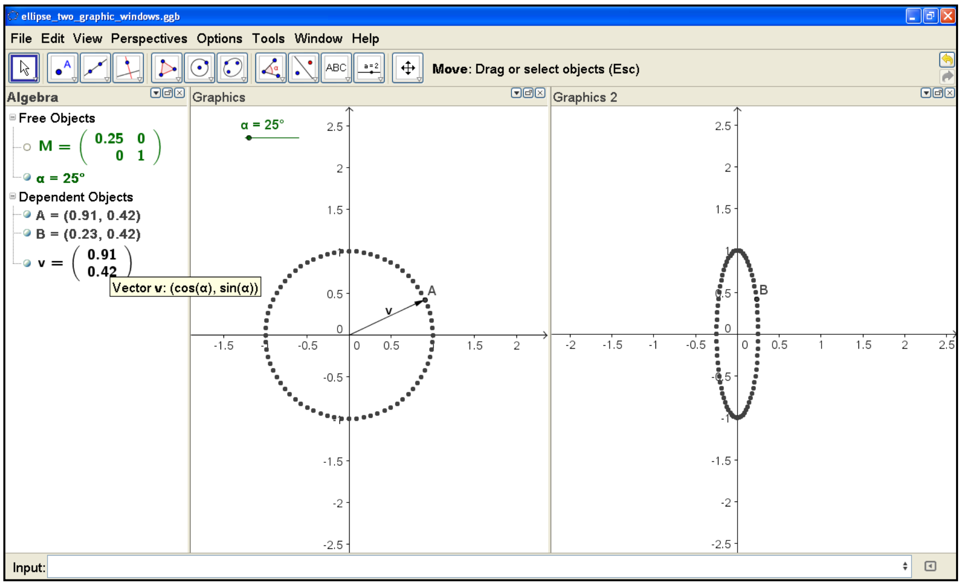


Fig. 10 A circle being traced out in Graphics view 1 and its image in Graphics view 2.

Define a slider α, a vector $v = (\cos(\alpha), \sin(\alpha))$ and a matrix $M = \begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix}$. Click on *Graphics 2* to activate it and write *M*v* in the input field. Define the points *A* and *B* as the endpoints of the vectors *v* and *M*v*, right click on them and check *Trace on*. Now move the slider α. The point *A* now traces out a circle in *Graphics 1* and the point *B* traces out an ellipse in *Graphics 2*.

We can generalize this construction by defining two sliders *a* and *b* and defining the matrix $M = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. For any fixed value of *a* and *b* the point *B* will now trace the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ when the slider α goes from 0 to 360. You can test this by defining the ellipse in *Graphics 2* (type the definition in the input field and make sure that it appears in *Graphics 2*).

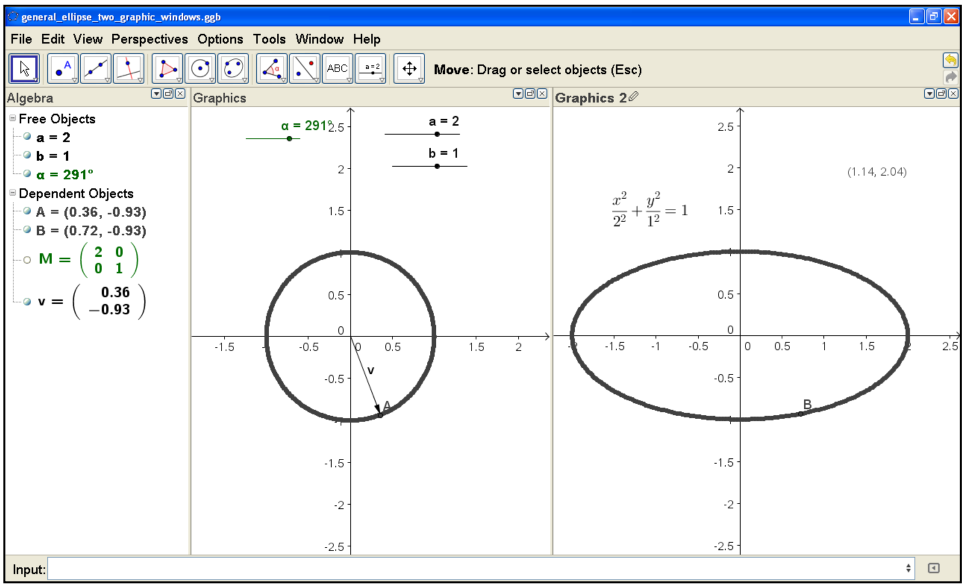


Fig. 11 The image of a circle under a transformation defined by a diagonal matrix.

Task: Investigate what happens if you change the matrix to non-diagonal matrix. Define two new sliders *c* and *d* and examine the matrix $M = \begin{pmatrix} a & c \\ d & b \end{pmatrix}$.

It is also possible to just define a circle c and then use the command *ApplyMatrix[M,c]* to get the ellipse in the second graphics view.

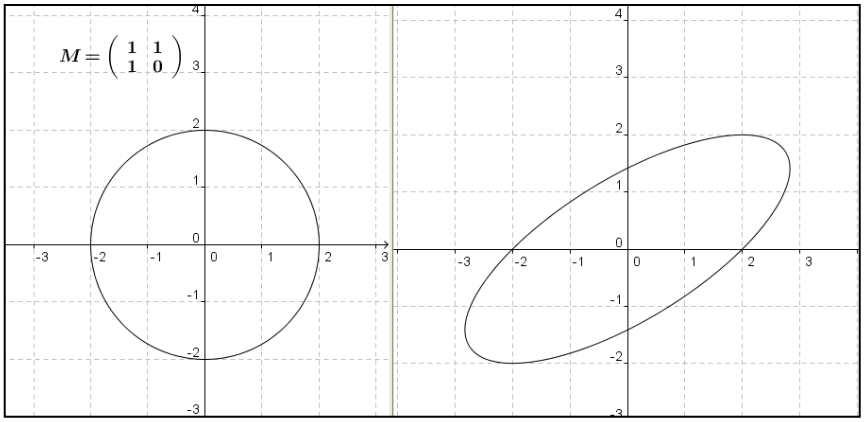
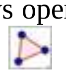


Fig.12 An ellipse as the image of a linear transformation

GeoGebra gives the equation of the ellipse and we can use the command *Focus* to get the foci. If we want to get the equation by hand we need to consider the following: if we use *u* and *v* for the coordinates in *Graphics 2* we have for the given matrix *M* that $u = x + y$ and $v = x$ and we need to write *x* and *y* in terms of *u* and *v*. The latter equation gives directly that *x* = *v* and *u* = *x* + *y* gives *y* = *u* – *x* = *u* – *v* so the equation of the circle $x^2 + y^2 = 4$ gives us $v^2 + (u - v)^2 = 4$ that is $u^2 - 2uv + 2v^2 = 4$. This can easily be checked by typing the last equation into the input field.

9 The area of the image of the unit square

We are going to study the effect of linear transformations on the unit square and the area of its image. We start by creating a worksheet with both graphics views open, define 4 sliders *a*, *b*, *c* and *d* and the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We then use the polygonal tool  to create a unit square with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$ in graphics view 1. Now we use the command *ApplyMatrix* to get the image of the unit square under the linear transformation defined by *M*.

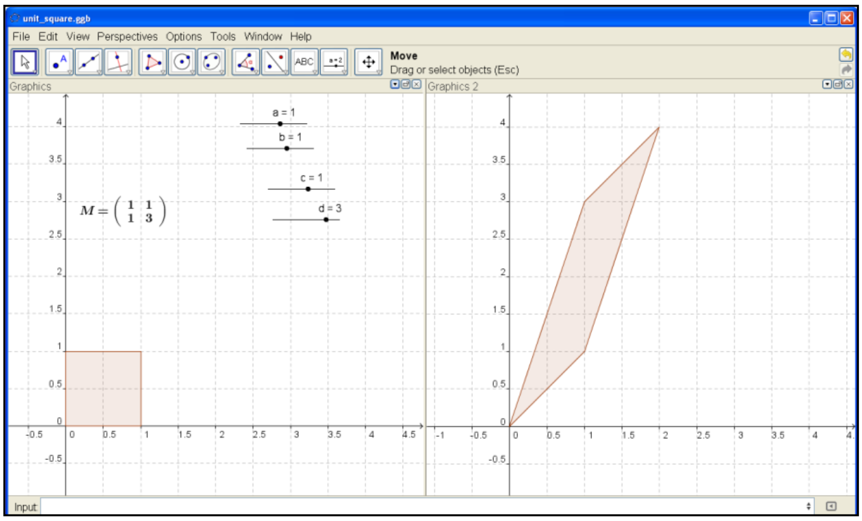


Fig. 13 Image of the unit square under the transformation defined by *M*.

We experiment with the matrix above and examine the area of the image. This is easier if we put the increments of the sliders equal to 1 of all sliders.

Task: Make the construction above and put the value of $a = 1, b = 0, c = 0$ and $d = 1$. What does the image of the unit square look like in this case and what is its area? Now change the value of *b* to 1. What is the area of the image? It is easiest to use the area tool  in GeoGebra.

Task: Make some further investigation on the area by changing the values of the sliders as given below and record the area in each case:

a) Keeping $a = 1, c = 0, d = 1$, put $b = -1, b = 2, b = 3$ etc. What is the effect of the value of b on the area of the image?

b) Put $a = 2, a = 3$ etc. and keep the other values fixed. How does the area of the image depend on a? What happens with the area if the value of a is negative?

c) Now do similarly for d and finally check the area for $a = 2$ and $d = 2$ as well as $a = 2$ and $d = 3$.

d) You might now have some theory concerning the effect of the values of a, b and d on the area given that $c = 0$. Now put a, b and d equal to 1 and check the area for the following values of c: $-2, -1, 0, 1, 2, 3$.

e) Write down a formula (in a, b, c and d) for the area of the image and test it for several values of these parameters.

10 Determinants and inverses

In the previous section your conclusion has hopefully been that the area of the image is $|ad - bc|$. This is the absolute value of the *determinant* of the matrix M defined as $\det M = ad - bc$ that is, the product of the diagonal elements minus the product of the anti-diagonal elements.

Task: Using your construction from before find nonzero values of a, b, c and d such that the determinant is 0. What does the image look like in those cases?

For a function f from $\mathbb{R}$ to $\mathbb{R}$ we sometimes have an inverse defined e.g. if $f(x) = 3x - 1$ we have the inverse $g(x) = \frac{x+1}{3}$ because $f(g(x)) = x$ and $g(f(x)) = x$. Similarly for functions $T: \mathbb{R}^2 \to \mathbb{R}^2$ we sometimes have inverses and these are particularly easy to study for linear transformations as they are themselves linear transformations and can thus be given by matrices as well.

In the case below we have applied the matrix M to the unit square S to get the polygon MS in graphics view 2. The matrix N has then been applied to the polygon MS which resulted in the polygon NMS in graphics view 1.

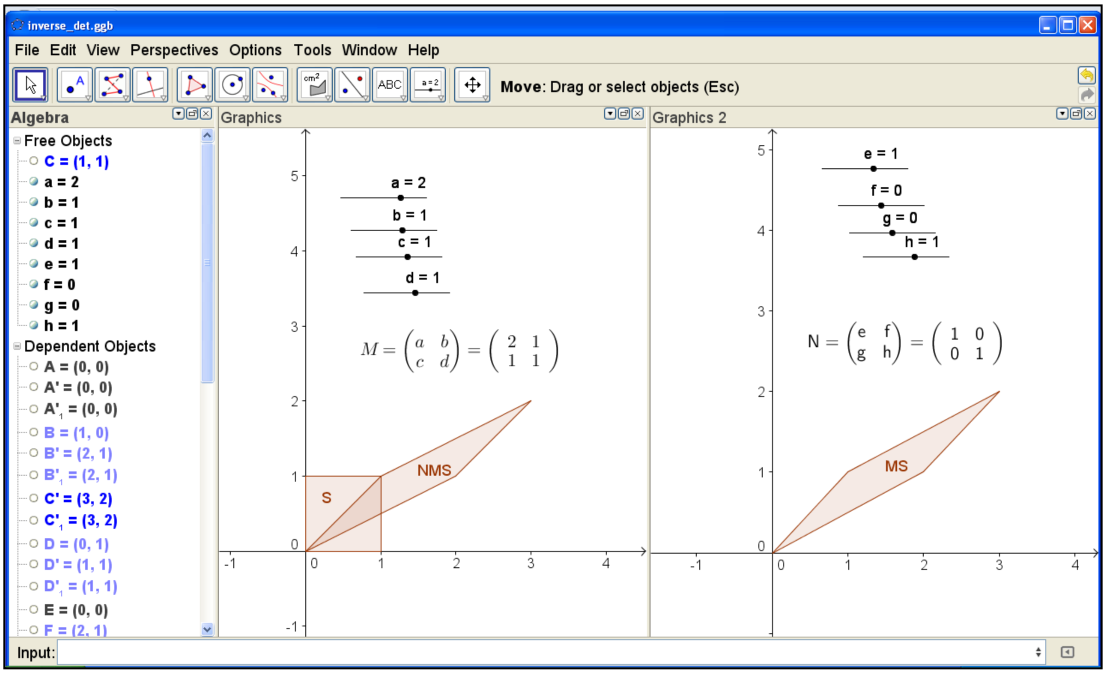


Fig. 14 Products of matrices

For the matrix *N* to be an inverse of *M* it needs to be such that *NMS* coincides with *S*.

Task: Create the construction above and find values of the sliders *e*, *f*, *g* and *h* such that *NMS* coincides with *S*. Make the increments of all your sliders equal to 0.5.

Hint: it might be easier to do this with say $a = 1, b = 1, c = 0$ and $d = 1$.

Task: Make a conjecture on the inverse of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Task: Calculate the determinant of the matrices *M* and *N* (there is a command called *Determinant* in GeoGebra).

Task: Change the values of *M* such that $a = 2,\ b = 1\ ,\ c = 0$ and $d = 1$ and find values of *e*, *f*, *g* and *h* such that *N* is the inverse of *M*. Does your earlier conjecture hold?

Task: Consider the determinants of *M* and *N* in the task above and formulate a theory for the relationship between the determinants of the two matrices. Test this out by changing the values of *a, b, c* and *d* and finding the inverse for this new matrix.

Task: Formulate a new conjecture for the inverse of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Test your conjecture by multiplying your conjectured inverse with *M*. Your answer should be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the identity matrix which is the matrix that maps the unit square to the unit square.

References

[1] Anton, H. and Busby, R. *Contemporary Linear Algebra.* NJ, USA: John Wiley and Sons Inc.

[2] GeoGebra, downloadable from http://www.geogebra.org.