# Investigating 2 by 2 matrices – part II

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## 1 Introduction

This chapter is a continuation of the study in *Investigating 2 by 2 matrices – part I*. We utilize the possibility of having two graphics windows in GeoGebra 4.0 to investigate the effects of transformations from the plane to the plane. Here we focus on the different types of transformations i.e. shears, rotations, reflections and compressions/dilations. For technical details concerning the use of GeoGebra see part I.

In the last two sections we study more advanced topics, i.e. eigenvalues and maps from the complex numbers to the complex numbers.

## 2 The unit square and the effect of linear transformations

We are going to study linear transformations through their effect on the unit square. We start by creating a worksheet with both graphics views open, define 4 sliders *a, b, c* and *d* and the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We use the polygonal tool to create a unit square in Graphics view 1 and the command *ApplyMatrix* to get its image under the linear transformation defined by *M*.
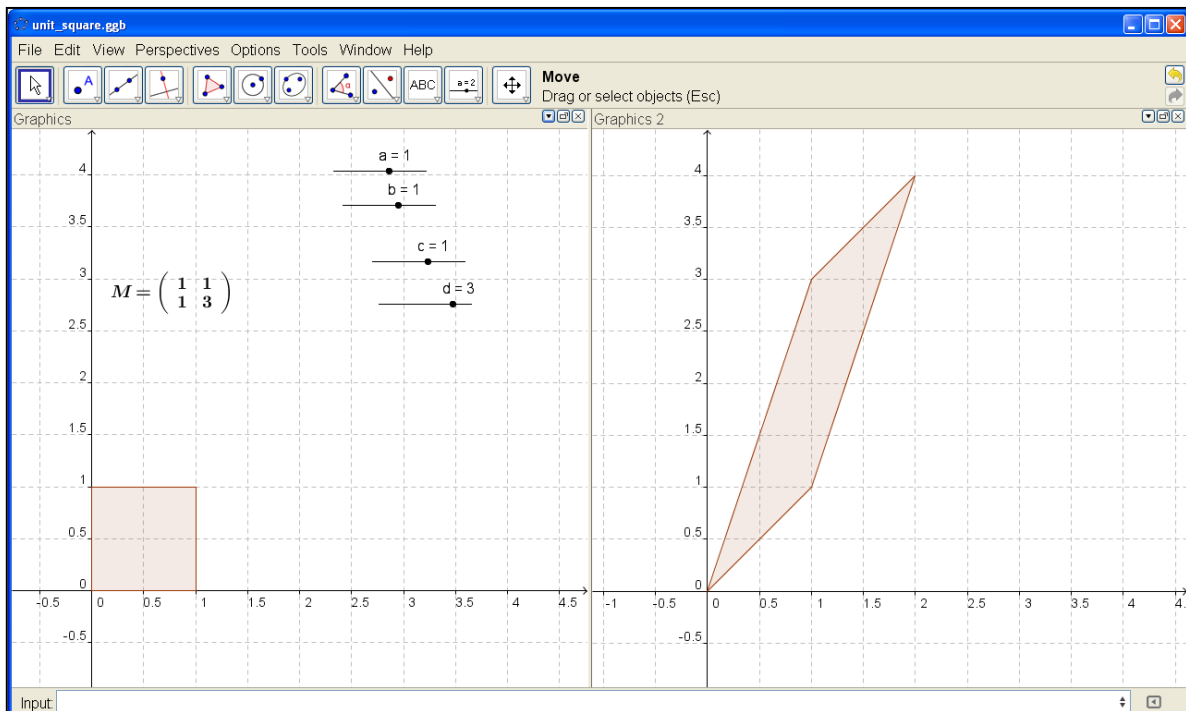


**Fig. 1** The image of the unit square under the transformation defined by the matrix *M*.

*Task:* Create the worksheet above.

We are going to use this to study the matrices of certain transformations such as rotations, reflections etc. These are given by tools in GeoGebra as seen below:
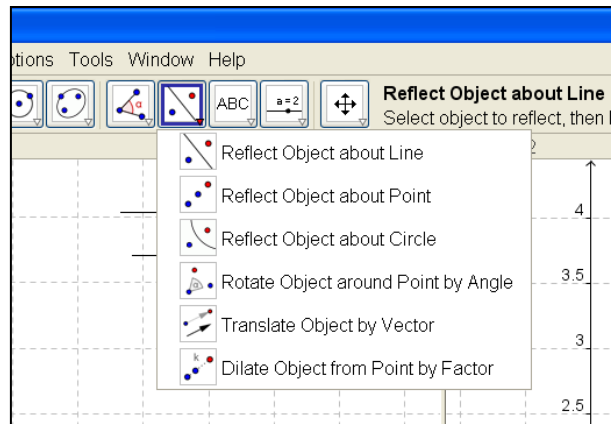
**Fig. 2** Built in transformations in GeoGebra.

*Task:* Use the dilation tool  to dilate the unit square by factor 2. Change the values of the sliders until the image of the matrix transformation is the same as the dilated unit square. You may want to move objects between Graphics views 1 and 2, this is done by opening *Object Properties* and then the tab *advanced*.

*Task:* Use the rotation tool,  to rotate the unit square 45°. Move the rotated square to Graphics 2. It might be informative to show the image in *both* views. Now change the values of the sliders defining the matrix until the image under the matrix transformation is identical to that under the rotation with the rotational tool.

*Task:* Draw a line through (0, 0) and (1, 1) and use the reflection tool  to reflect the unit square in this line. It will reflect into the unit square again but the corner point (1, 0) is reflected to (0, 1) and vice versa. Find the matrix that gives this reflection.
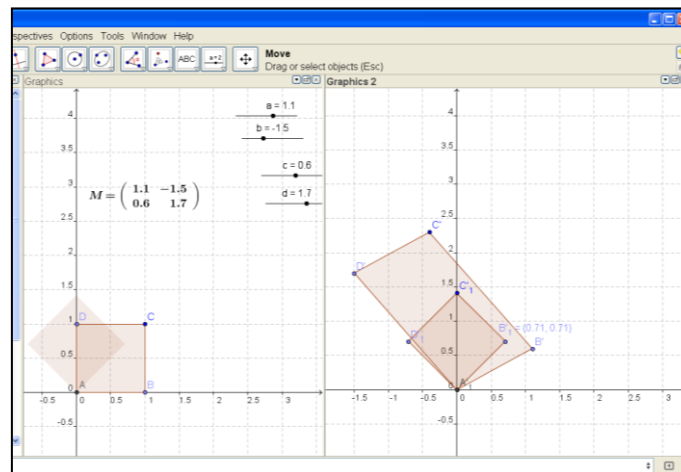


**Fig. 3** The unit square rotated 45° and its image under the transformation defined by *M*.

# 3 Shears and compressions

The linear transformation given by a matrix of the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ (or $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$) is called a *shear* in the *x*-direction with factor *k* (or a *shear* in the y-direction with factor *k*).

*Task:* Make a worksheet with both graphics views open and define the unit square in *Graphics 1*. Define a slider *k* and the matrices above. Apply the matrices to the unit square and observe the effect of changing the value of *k*. What is the area of the image?
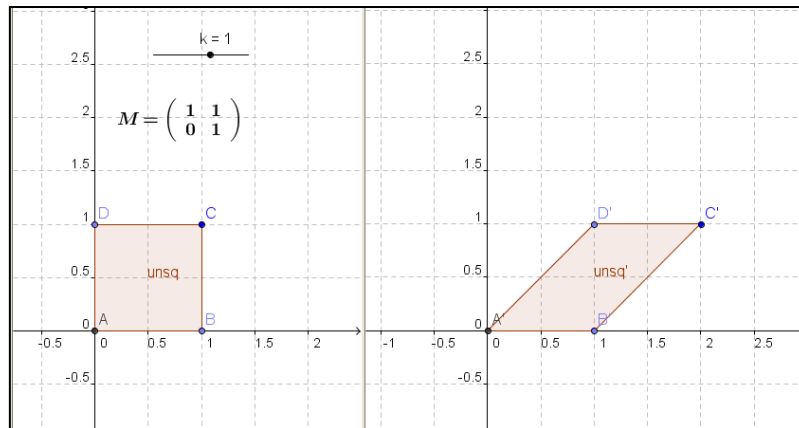
**Fig. 4** The image of the unit square under a shear.

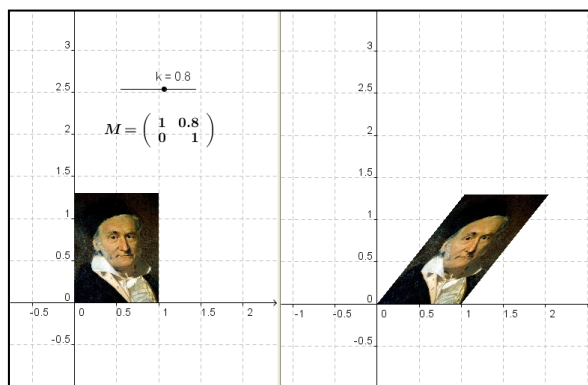*Task:* Repeat the task above for compressions and expansions in the *x* and *y* directions. These are matrices given by $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$.

We can import photographs into GeoGebra using the tool ![icon]. After this is done we can use the command *ApplyMatrix[matrix, picture]* to apply transformations to photographs.



**Fig. 5** Photograph from Wikipedia Commons,
http://en.wikipedia.org/wiki/File:Carl_Friedrich_Gauss.jpg

*Task:* Try this out with your favorite photograph.

# 4 Exercises

Create a worksheet with both graphics views open. Define a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ using four sliders *a, b, c* and *d* and create the unit square in Graphics view 1. Mark the corner points are *A, B, C* and *D*. Apply the matrix to the square and make its image appear in Graphics view 2. Use the sliders to solve the following exercises from [1] (page 293).

Find the matrix for each of the transformations given below. Compute the determinant of the matrices, either by hand or by using the command *Determinant* in GeoGebra, and compare the result with the area of the image in Graphics view 2.

- a) Contraction with factor $\frac{1}{5}$.

- b) Compression in the x-direction with factor $\frac{1}{3}$.

- c) Expansion in the *y* - direction with factor 6.

- d) Shear in the *x* – direction with factor 3.

- e) Dilation with factor 5.

f)   Expansion in the $x$ – direction with factor $3$.

g)   Compression in the $y$ – direction with factor $\frac{1}{6}$.

h)   Shear in the $y$ – direction with factor $2$.

Describe the geometric effect of the transformation defined by the given matrices:

a) $\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$   b) $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$   c) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$   d) $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

Compute the determinant of the matrices in a) and b).

# 5 Compositions of transformations

If $T$ and $S$ are linear transformations $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ then the composition of $S$ and $T$ is the transformation $S \circ T = S\big(T(u)\big)$ where $u$ is an element in $\mathbb{R}^2$. If $N$ is the matrix of $S$ and $M$ is the matrix of $T$ then the matrix of the composite transformation is $NM$, that is the product of the two matrices.

To visualize this it would be nice to have 3 graphics views but this is not possible in the current version of GeoGebra.

We define two matrices, $N$ and $M$, using 8 sliders and apply those to the unit square such that the image appears in Graphics view 1. We then apply the matrix $M$ to the image of the unit square under the transformation given by $N$ and make that image appear in Graphics view 2. We do similarly for the other matrix.
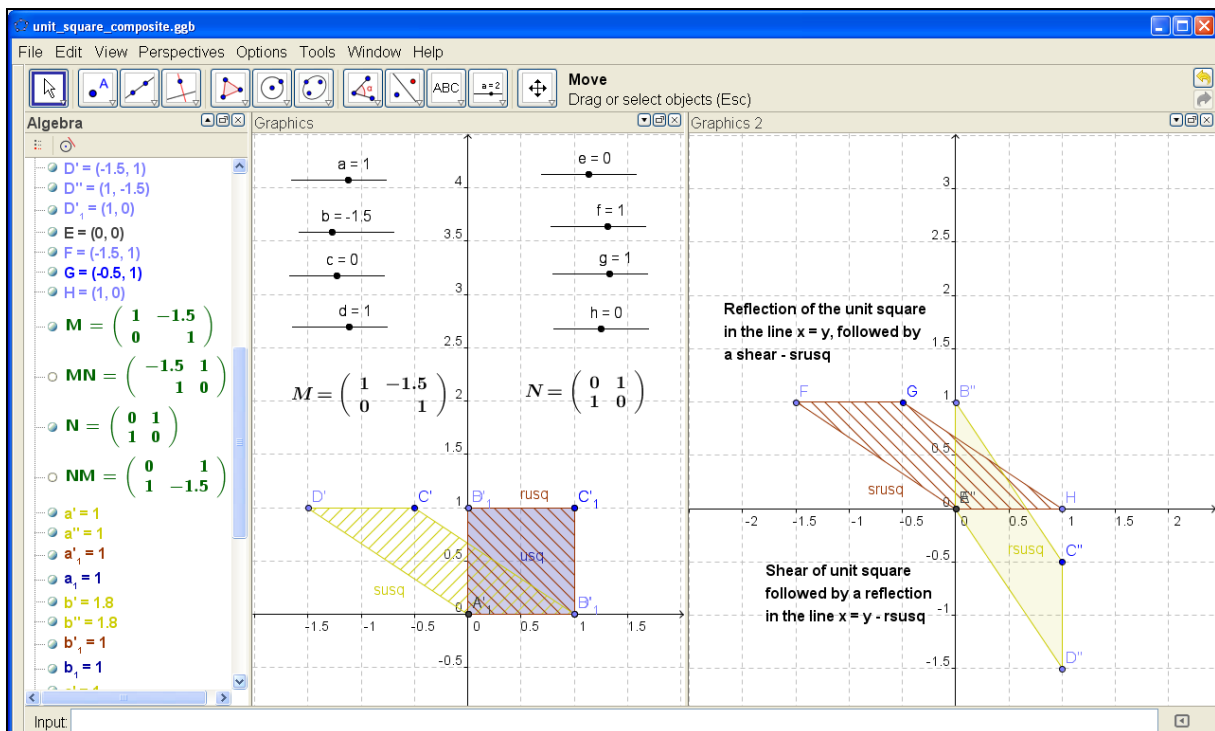


**Fig. 6** Compositions of transformations

We see that the images, in *Graphics* 2, of the unit square under the composite transformations $MN$ and $NM$ are not the same. In the algebra view we can see that the two products $MN$ and $NM$ are not the same.

*Task:* Construct the worksheet above. Experiment with the sliders. Are there any values of the sliders that give the same images in *Graphics* 2?

We see from the above that there are many possibilities of composing the simple linear transformations we have studied. Conversely, one might ask if it is possible to write any transformation as a composition of the simple ones? This is indeed the case as the following theorem states [1]:

**Theorem 6.4.4** If $A$ is an invertible matrix, then the corresponding linear operator on $\mathbb{R}^2$ is a composition of shears, compressions, and expansions in the directions of the coordinate axes, and reflections about the coordinate axes and about the line $y = x$. (p. 311)

We consider the matrix $L = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ and try to write it as a composition of the transformations mentioned in the theorem above.
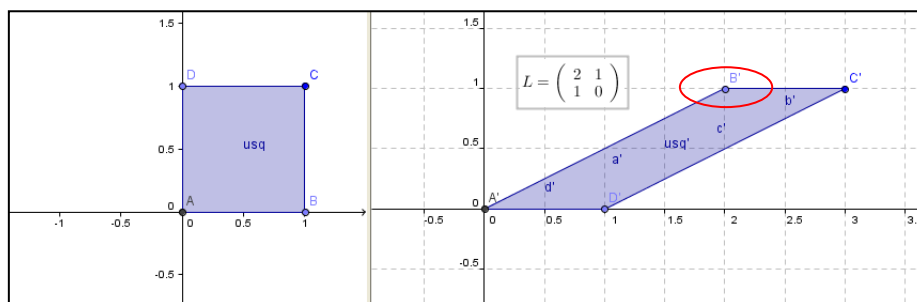


**Fig. 7** The image of the unit square under the transformation defined by the matrix $L$. Marked in red is the point $B'$ which is the image of the point $B$.

From the picture the transformation appears to be a shear in the $x$ – direction with factor 2. However if we apply such a shear to the unit square we get the image below.
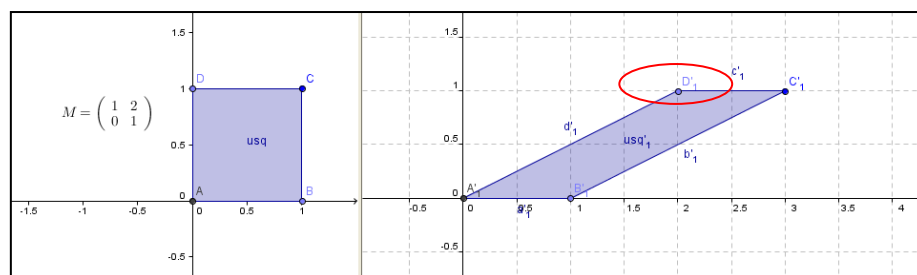


**Fig. 8** Marked in red is the vertex $D'_1$ which is the image of $D$.

This is the same parallelogram as before but some of the vertices are not the same, e.g. the vertex in the upper left corner is the image of the vertex $D$ but in the former picture we saw that it was the image of the vertex $B$. So it seems that we need to switch the points $D$ and $B$ before applying the shear transformation which is exactly what a reflection in the line $x = y$ would give us. We therefore define the matrix $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is a reflection in the line $x = y$ and apply it before applying the shear.
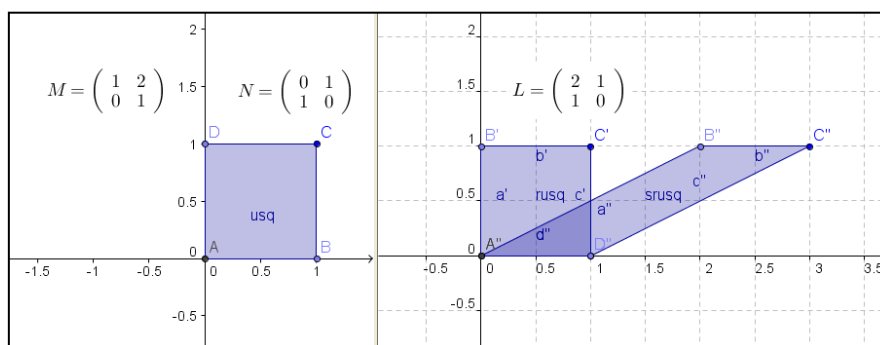


**Fig. 9** The matrix $N$ is applied to the unit square (*usq*), this gives the unit square *rusq* which we then apply $M$ to and get the parallelogram (*srusq*).

The parallelogram above is exactly the parallelogram we get when we apply the matrix $L$. We can now use matrix multiplication to verify that $M*N = L$.

For more complicated matrices this is not so easy. We consider the matrix $L = \begin{pmatrix} 1 & -3 \\ 4 & 6 \end{pmatrix}$ and write it as a composition of compressions, expansions, reflections and shears. We solve this algebraically by using elementary row operations as demonstrated below:

$$L = \begin{pmatrix} 1 & -3 \\ 4 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 \\ 0 & 18 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Arrow 1: we multiply the first row by 4 and subtract from the second row. This is amounts to multiplying the matrix $L$ by the elementary matrix $E_1 = \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}$ so the second matrix is $E_1L$.

Arrow 2: divide the second row by 18 which is the same as multiplying by the elementary matrix $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{18} \end{pmatrix}$ so the third matrix is $E_2E_1L$.

Arrow 3: multiply the second row by 3 and add it to the first row which is the same as multiplying by the elementary matrix $E_3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

We thus have: $E_3E_2E_1L = I$ so $L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 18 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ which gives that the transformation defined by $L$ is composed of:

- A shear in the negative $x$ – direction with factor 3
- An expansion in the $y$ – direction by a factor 18
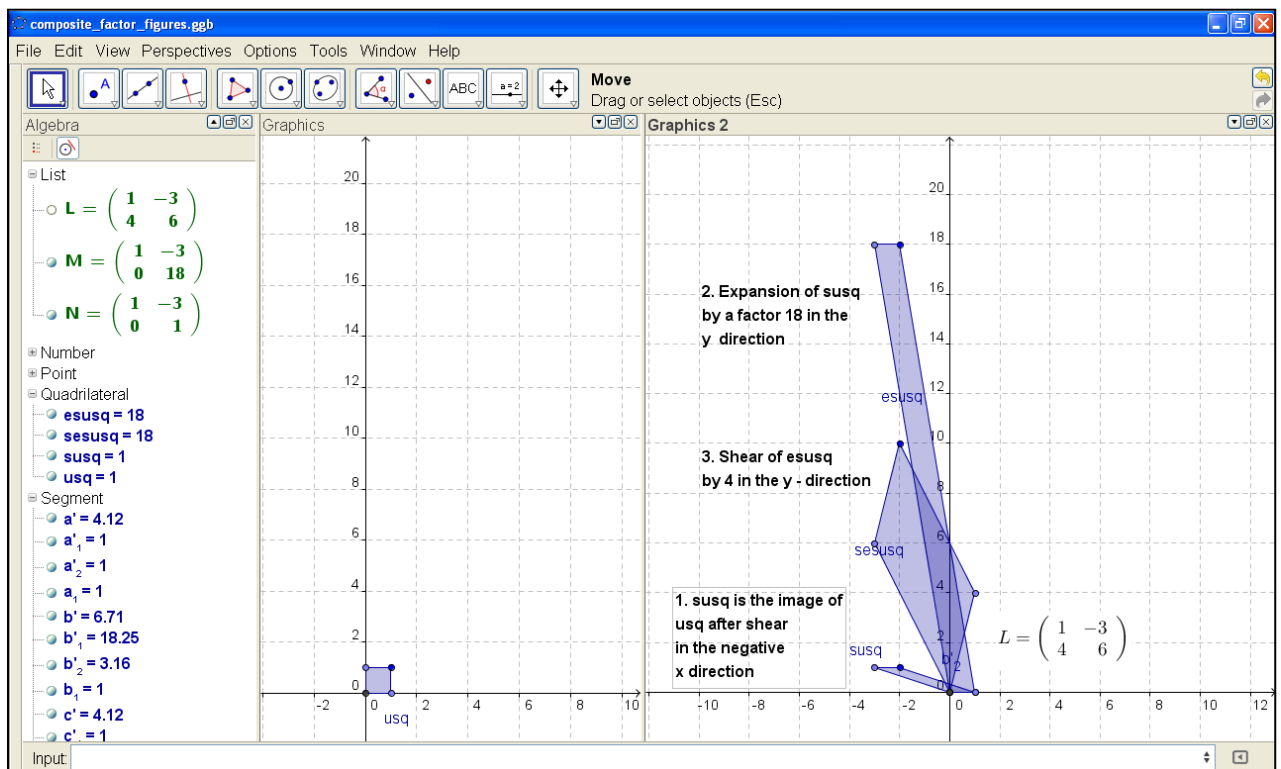- A shear in the $y$ – direction with factor 4.



**Fig. 10** The transformation $L$ given as a composition of simple transformations.
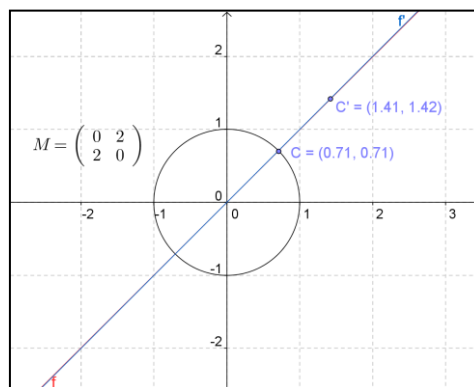
# 6 Eigenvalues and eigenvectors

We start by stating the definition of an eigenvector and an eigenvalue ([1], page 211):

If $A$ is an $n \times n$ matrix, then a scalar $\lambda$ is called an *eigenvalue* of $A$ if there is a nonzero vector x such that $Ax = \lambda x$. If $\lambda$ is an eigenvalue of $A$, then every nonzero vector $x$ such that $Ax = \lambda x$ is called an *eigenvector* of $A$ corresponding to $\lambda$.

We now demonstrate how we can visualize eigenvectors of 2 by 2 matrices in GeoGebra. We create a worksheet with a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where *a, b, c* and *d* are sliders like before. We use the circle tool to create the unit circle and the point tool to get a point $C$ on the unit circle. We use the line tool to define a line *f* through (0,0) and *C*. We then use the command *ApplyMatrix[M, f]* to get the image *f'* of the line *f* under the transformation and *ApplyMatrix[M, C]* to get the image of *C*. Everything is now taking place in *Graphics 1* so we therefore choose different colors (red and blue) for the two lines to distinguish them. We now use the moving tool to move the point *C* along the unit circle. This moves both the red and the blue lines.
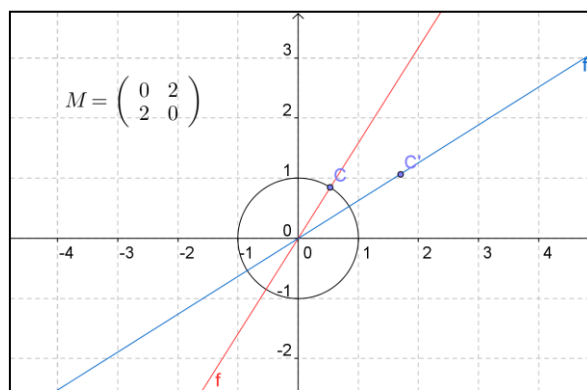


**Fig. 11** The blue line is the image of the red line.



**Fig. 12** OC is an eigenvector of the linear transformation.

When the two lines coincide we have that the vector $OC$ is mapped to a scalar multiple of itself so we have an eigenvector. To find the eigenvalue we divide the length of $OC'$ by the length of $OC$.

*Task:* Create the worksheet described above and test this for the matrices given in the exercises earlier. Do shears and rotations have any eigenvectors?

# 7 Complex transformations

We can study maps from $C$, the complex numbers, to $C$ using similar methods as were shown in *Investigating 2 by 2 matrices – part I* to study transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. For instance, say we want to study the map

$$z \longrightarrow \frac{z+2}{z-1}.$$

We open a worksheet with two graphics windows and define a line and a point *F*, on the line. We right click on *F* to open *Object Properties* and under the tab *Algebra* we select *Complex Number*.

We then click on *Graphics 2* and write z = (F + 2)/(F - 1) in the input field. We right click on *z* and put *Trace on* and then we move *F* along the line.
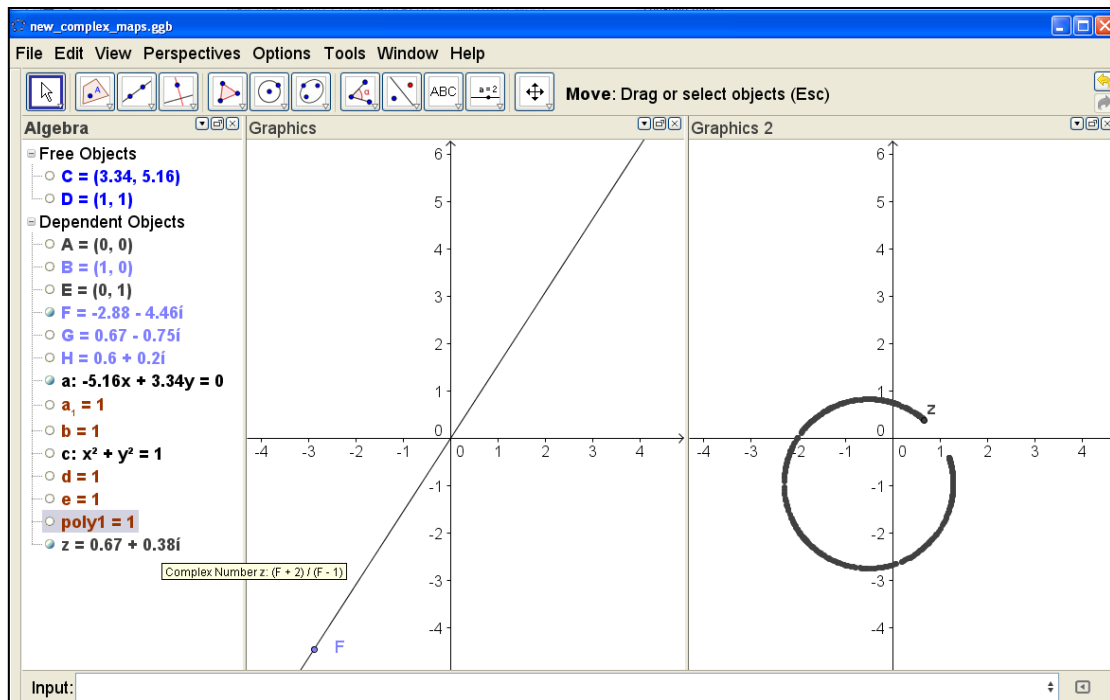


**Fig. 13** Image of a line under the complex map $z \; \longrightarrow \; \dfrac{z+2}{z-1}$

We then also define points on the unit circle and on the unit square, define their values in *Graphics* 2 using the same definition as before. Below the traces of this are shown.
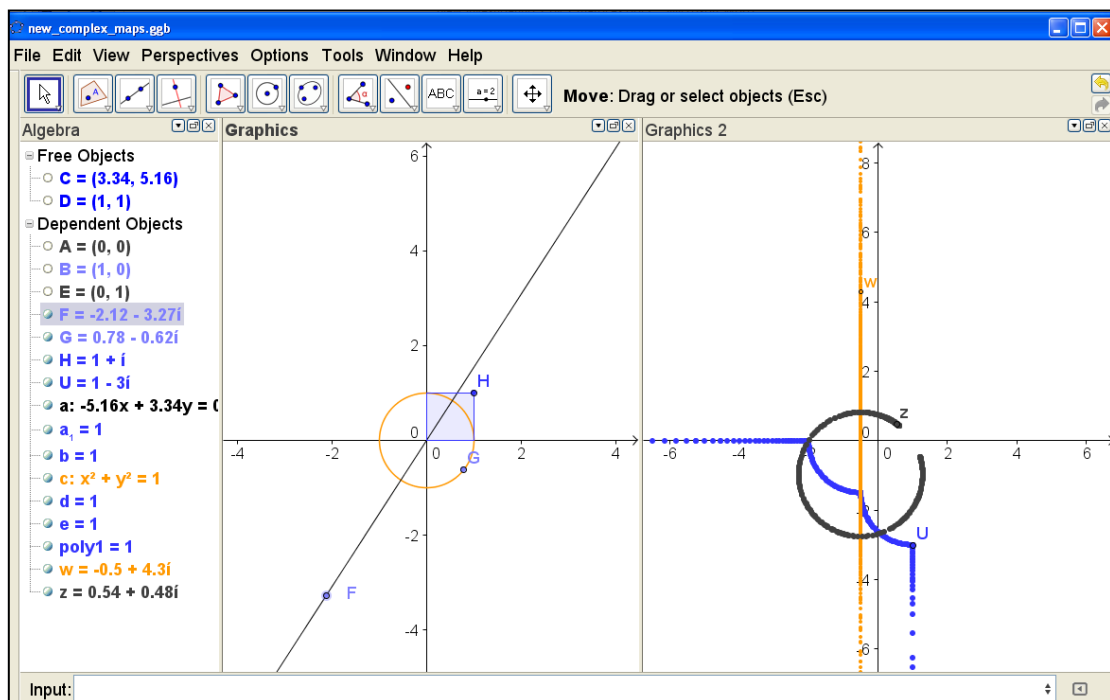


**Fig. 14** The image of a line, a circle and the boundary of the unit square under the complex map
$$z \; \longrightarrow \; \dfrac{z+2}{z-1}$$

*Task:* Investigate the complex maps by the methods above

a)  $z \longrightarrow \dfrac{z-i}{z+i}$

b)  $z \longrightarrow z^2$

c)  $z \longrightarrow z + \dfrac{1}{z}$

# References

[1]   Anton, H. and Busby, R. *Contemporary Linear Algebra.* NJ, USA: John Wiley and Sons Inc.

[2]   GeoGebra, downloadable from http://www.geogebra.org.