# Best spot - investigation with circles

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## 1 Introduction

Bratislava, the capital of Slovakia, is little big city on river Danube with many bridges. One of the tourist attractions is also lookout at these bridges. On the Fig. 1 we have a top view of three of these bridges. There are of different ages. On the left side there is the oldest bridge which is also called so, the Old Bridge. In the middle there is the newest bridge called the Apollo Bridge. The name of the last one is the Harbour Bridge.


**Fig.1**: Top view of three bridges in Bratislava.

As a common tourist you can walk along the Danube promenade which we have coloured in red. This promenade is about 455 meters long and there are three viewpoints. As an uncommon tourist you can ask yourself, where is the "best spot" to look at the Apollo Bridge Fig 2.

**Fig 2**. Apollo Bridge (Wiki media)

We can come to an agreement that the “best spot” is characterized by the biggest angle under which the Apollo Bridge is seen. There are three view points (1, 2, 3). Which one is the best, according to our agreement, for observing the Apollo Bridge?

As you identify yourself as an uncommon tourist that it means you are close, more than you thought, to a scientist or rather to a problem solver. Process that we need to go through is called “inquiry process” and it is similar to a method how mathematicians work.

Are you ready? Let’s start.

# 2 Problem 1

How it is possible to find an answer to previous question? Where on the promenade is the “best spot” to see the Apollo Bridge?

## 2.1 Possible approach

We can simplify the problem with a model. It is based on the fact that the bridge touches the bank of the river. So we extend the line along the promenade and construct a line perpendicular to the bridge. It means that all places along the promenade have one common ray and we just move the vertex of view angle closer to the bridge. Then it is not difficult to understand that the closer the vertex is to the bridge, the bigger the angle is. What about the points that are very close to a side of bridge? What angle does it approach (converge) to? Try to look out on the internet what is the most convenient angle to look at an object.

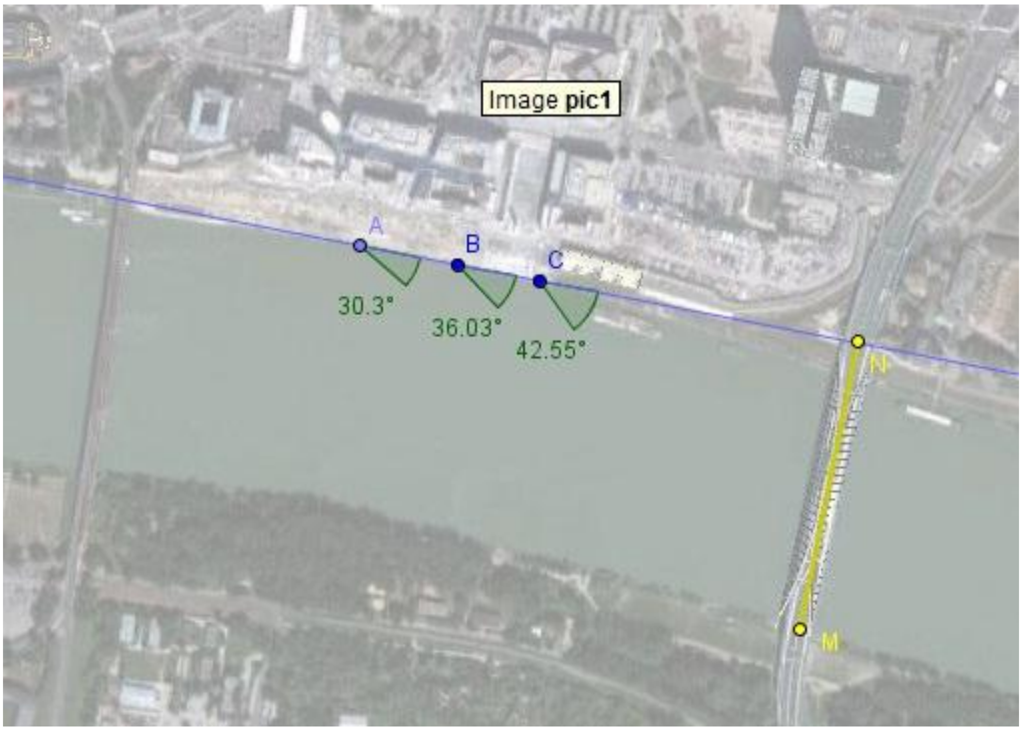



**Fig.3** Lookout at the Apollo Bridge from three view points on Danube promenade. (Photo by Google Earth)

**Fig. 4** View point C **Fig. 5** View point B **Fig. 6** View point A

## 2.2 Teacher´s notes

We can easily let the students investigate and let them come up with their own solutions. It is appropriate to give them the photocopy of the map or let them explore the problem in one of the geometrical software. The top view of the situation is easily available from the freeware software Google Earth. Possible geometrical software can be for example Geogebra. It is important to encourage students formulate their own ideas and further questioning that may help them find a solution. It is important not to tell the solution immediately. We can also ask students to formulate their explanation and let them write it down on a sheet of paper.

## 2.3 Development of mathematical concept

Comparison of angles

Mathematical modelling

# 3 Problem 2

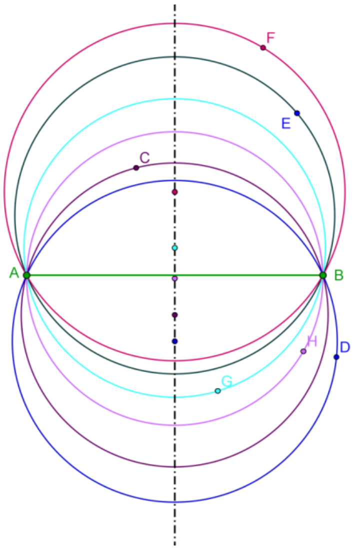
Many tourists like to experience the Danube cruise. So do us. On this cruise we have again the opportunity to see the Apollo Bridge from the different perspectives. What is the “best spot” to look at the Apollo Bridge if we come from the Old Bridge?

**Fig. 7.** Old Bridge (in the front) and the Apollo Bridge (in the back)

## 3.1 Possible approach

The key question is how we can find the biggest angle that hold two points. We try again to develop (use) mathematical model in which some limitations of human eye and other factors are not considered. After that we try to look at the problem more realistically. Viewing angle is the maximum angle at which an object can be viewed with acceptable visual performance. The size of this angle increases when the vertex approaching to the center of the segment *MN*. Let’s call this center *S*. From point S there is the same distance to points M and to point N.

But within the mathematical model we keep thinking about other places which are equally distant from both endpoints. We are talking about infinite numbers of these places. We would call them circles passing through points A and B.



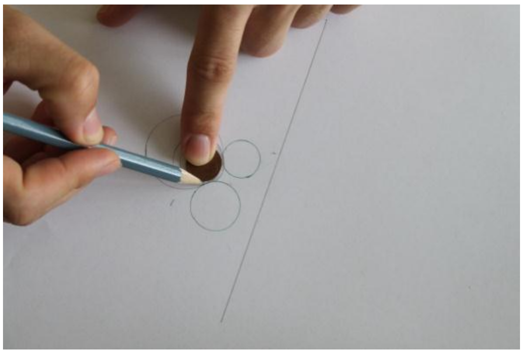



**Fig.8** Perpendicular bisector made in Geogebra.  **Fig 9.** Perpendicular bisector on a sheet of paper.

So the second key point is to find the locus that the centres of circles passing through end points of the Apollo Bridge M and N trace.

## 3.1.2 Investigation

We can use Geogebra software and tool “compass” where we set the radius along and then approach the circle to points M and N. The centres of the circles form a locus that we try to define after investigation. It is appropriate to have at least three circles.

Through the inductive process of locus identification we can define it. The locus of points that is equally distant from two points is perpendicular bisector.

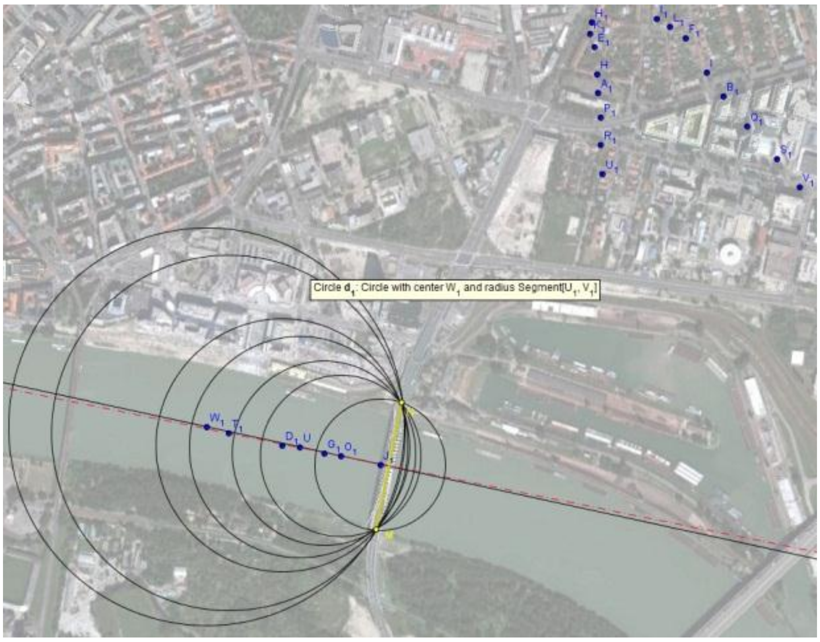


**Fig 10.** Perpendicular bisector made by tool compass and tool perpendicular bisector, red line, on the map in software Geogebra. (Photo by Google Earth)

We can continue also with the points different from the perpendicular bisector.

We can develop the idea with the midpoint of the segment. Let the students investigate the angles in circle that has the centre in midpoint of the segment *MN*. Let call this circle $k_1$. What is the angle *MSN*? By now we can choose any point, for example *X* on the circle. How big is this angle? What about the other points? How can we define the central angle and relationship with periphery angle? The measure of the periphery angle is the half of the central angle in the circle.

Based on this knowledge and the information about convenient angel from problem 1, we can identify the places form which we can see the bridge in this angle. Within our investigation we would like to find out how big the angle β is in comparison with central angle. The size of this angle increases when the point *S* approaching along the axis of the segment *MN* to its center. Then the circle *k* has to have with the line *c* at least in one common point. The maximum allowable approaching of the point *S* to the line *MN* occurs when the circle touches the line *c* in only one point *X*. Then will be the angle *MSN*and therefore the angle *MXN* maximal.

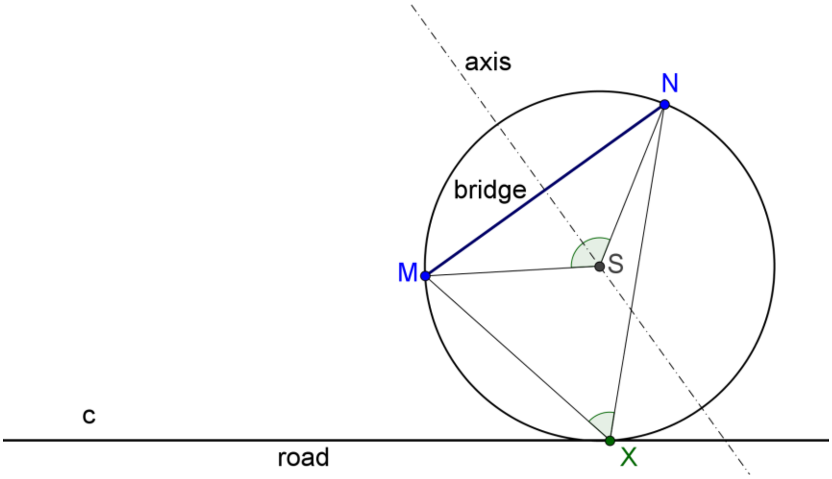



**Fig 11.** Centre and periphery angle

## 3.2 Teacher´s notes

The process of investigation can sometimes lead into a dead end. It is necessary to be prepared to support the students' inquiry process with other starting points or within the key points that will encourage their further work.

## 3.3 Development of mathematical concept

Mathematical modelling

Perpendicular bisector -locus

Relationship between centre and periphery angle

Circumscribed circle to a triangle

# 4 Problem 3

The Harbour Bridge is not as nice as the Apollo Bridge. But it is the bridge that carried express highway across the river Danube. This place is the first gateway for travellers by cars or buses from the D1 highway to see the old city. Once more time, which is the best spot from the Harbour Bridge to see the Apollo Bridge?

## 4.1 Possible approach

One of possible approaches is usage the concept of locus. Another one is to use theory of power point to the circle. We focus on the first approach.

We can make some arrangements and use Geogebra to model the situation. We put the Google Map of bridges in Geogebra and investigate this problem in this environment. At the beginning we construct the line *c* on one side of the bridge and a point *X* on this line. Than we press the angle button in Geogebra and measure the angle *MXN*. What if the point *X* (lying on line s) can move along line *c*? Then move with the point *X* along the line and observe what will happen with the angle. Can you explain why?

**How can we find the place where the angle is the biggest?**
How is the view angle connected to the distance from the object? Where is the place on the Harbour Bridge with the biggest view angle depending on the distance from the object – Apollo Bridge?

Imagine for example any point in the river. How is it connected to the circles with centres lying on the perpendicular bisector? We have investigated that in problem 2. From problem 2 we have gained the knowledge that the centre of circle lies on the perpendicular bisector, but we don't exactly know where. We are looking for the centre $S_x$ of circle $k_x$ that will touch the line s and pass through the points M and N.

**How can we find the place where the angle is the biggest?**
We can let students to give some ideas or use some prior suggestions. For example let the students investigate with circle in Geogebra, try to approach the circle to points *MN* and line *c*. Than we can discuss with students that it probably satisfies our purposes, but it is not mathematically exact.

Let backtrack – to move our thinking from the final result backwards. Such steps will lead us to a solution. We need to investigate the final solution. We need to look for a locus that has the same distance of line *c* and points *M* or *N*. **What is the locus equally distant from point *M* and also from line *c*?**

Metathinking and metacomments are very important in the process of investigation.

We can divide the problem into two problems.

A) Which locus do the centers of the circles touching two fixed points describe?
B) Which locus do the centers of the circles touching a fixed line and passing through a fixed point describe?

A) We investigated this situation in problem 2.

B) At the beginning we use inductive approach to better understand the concept. At first we need to think when is the view angle the biggest?

We can place circles on the plane touching the line and a point.

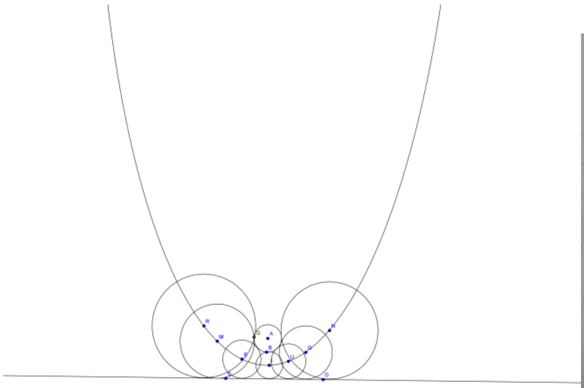


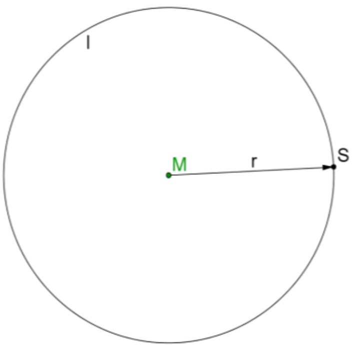
**Fig. 12** Investigating parabola in Geogebra **Fig. 13** Investigating parabola on piece of paper

To define the wanted locus we would like to use knowledge that we already have gained about circles and geometrical construction. (This approach is for experienced mathematics students able to apply their knowledge into a new situation to investigate a new concept.)

Now we can move from the environment of our map to universal model, because we don't exactly know where the point *S* (centre of wanted circle) is located.

1: We choose one point and call it point *M* that is given in the plane. (We cannot move it. In reality it is one of the endpoints of our bridge.) What is the set of points which are equidistant from a given point?

It is circle with centre in point *M* and constant distance *r* - radius.



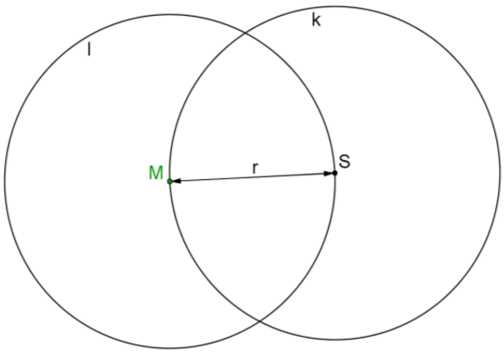




**Fig. 14** Investigating parabola step one **Fig. 15** Investigating parabola step two

Now imagine one point from the set which can be freely moved round the circle. Let choose one point and name it point *S*. This point lies on circle *l*. Then also point *M* lies on the circle *k*, which has centre in point *S*.

Now add to our thoughts line *p*. This line is tangent to the circle in point T which is different from the point M and lies on the circle *k*. Because of the tangent p to the circle *k* we know that ST is perpendicular to the line *p*. From the definition of the circle that we were talking about by the beginning of our thoughts, we also know that SM = ST.

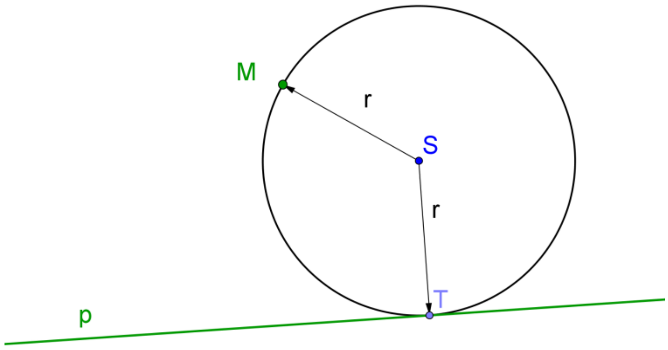



**Fig. 16** Investigating parabola step three

What is important is that point T is not fixed and so we can freely move with him along the line p. On the other side, point M is fixed.

What can we say about the distance from the point S to M and the distance from the point S to line p when we move the circle (we move it by the point T)? Move the point T and try to predict the results. Observe what will happen.

M_S_p.html

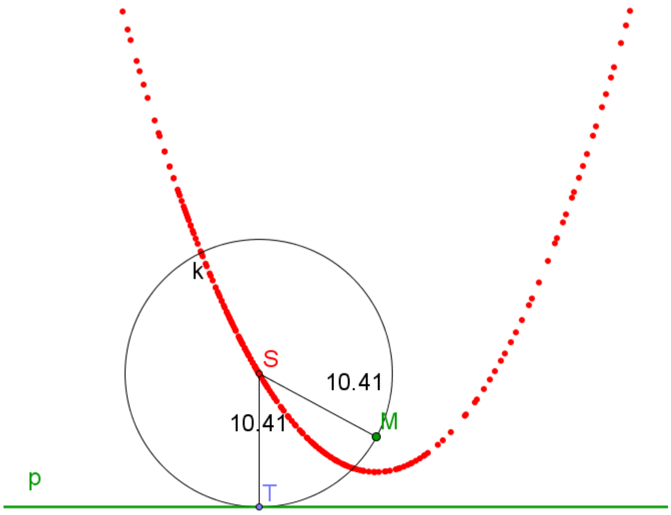



**Fig. 17** Conjecture about the locus line and point.

Imagine that point S trace red points. Once again what is the name of this locus?

- This curve is parabola. : it is the set of all points in the plane whose distance from a fixed point *M* is equal to their distance from fixed line *p*.

How many circles are there that touch point M and line p and they radius is r? How many points are on line?

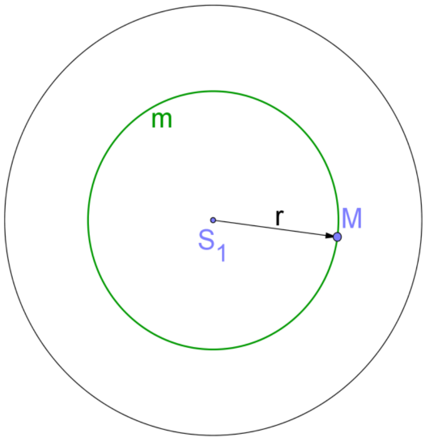
Till now we have considered that the point M is fixed in a plane. What will happen if we let the point M move along the circle, for example circle *m* with centre J?

Let have a line p and circle m with centre J. Imagine the point with the same distance from this line and circle *m*. How many points like this do there exist? What kind of locus do they define? Again go step by step:

Now imagine that the point *M* can move freely through the given circle *m* with center $S_1$.

- What is the set of points which are equidistant from a given circle?

It is a concentric circle.



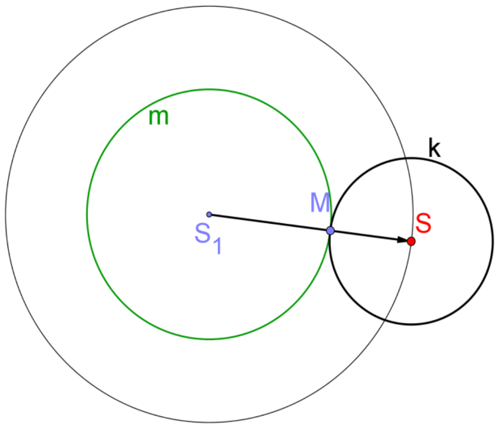




**Fig. 18** Investigating parabola-step 1 with circle **Fig. 19** investigating parabola -step 2 with circle

This concentric circle is also set of all centres of circles touching given circle *m.* All these centres are equidistant from given center $S_1$.

Hence the task to find set of all centers of the circles touching a fixed line and fixed circle is the same as to find the set of all centers of the circles touching a fixed line and passing through a fixed point.

parabola_aplet_k_p.html

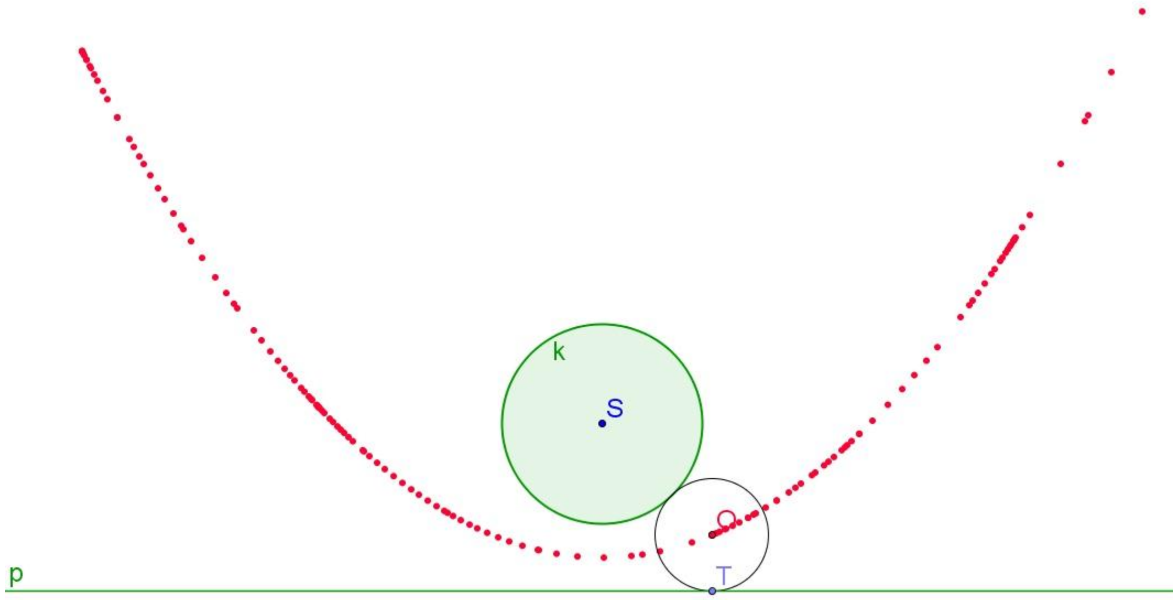



**Fig. 20** Conjecture about the locus line and circle.

Now we should be ready to move back to our map and find our best spot. We will construct two loci the perpendicular bisector to segment MN and parabola to line *s* and point M. The intersection of these curves gives us the centre of circle *k*, that is circumscribed to points M, N and line *s*. The intersections of this circle and line *s* give us the wanted “best spot” to look at the Apollo Bridge from the Harbour Bridge.

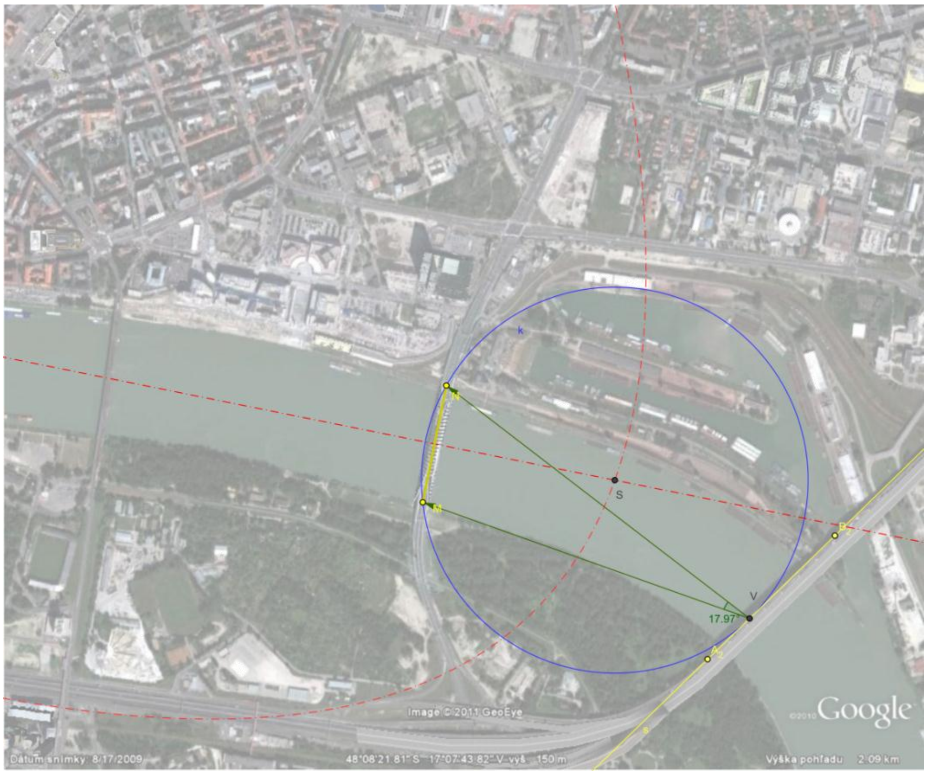



**Fig. 21** Application of gained knowledge for solution of problem 3

## 4.2 Teacher´s notes

This problem is suitable for year 10and 11 (1st and 2nd year of upper secondary school). Some students can have difficulties to come up with the solution by themselves. This may increase if students are not used to work this way. Even though the difficulties we should let students try and make some suggestions. There should be also some time for trying to make own approaches.

After some failure we can set up inquiry environment with lower level of openness that will help students to break the problem into smaller parts and understand the key concept.

This approach can help students in better understanding of usage of conic curves. It can also help them to use approaches, which was impossible without dynamic geometric software.

In the end we ask students to organize their results and let them formulate some model (in deductive way) that will work in different environments and with different objects. This approach may help them to see how and from where do the math theorems come from.

## 4.3 Development of mathematical concept

Parabola – locus

Mathematical modeling

Application of model, development of knowledge in finding out the solution of a real life problem

Inductive thinking

Making conjectures

# 5 Final problem

If we consider that the convenient view angle is 36 degree than we can generalize our solutions from problem two to a locus within which we can see the Apollo Bridge under this angle.

Again we can let the students to give their own ideas.

We choose the construction of locus within which we see the bridge under the angle 36 degree by calculation of angle in isosceles triangle. This way we can calculate the angle MSN that should be 72 degrees. From this point we make a circle. We do the same to other side. After that we have a set of points with the optimal view angel to the Apollo Bridge in Bratislava.

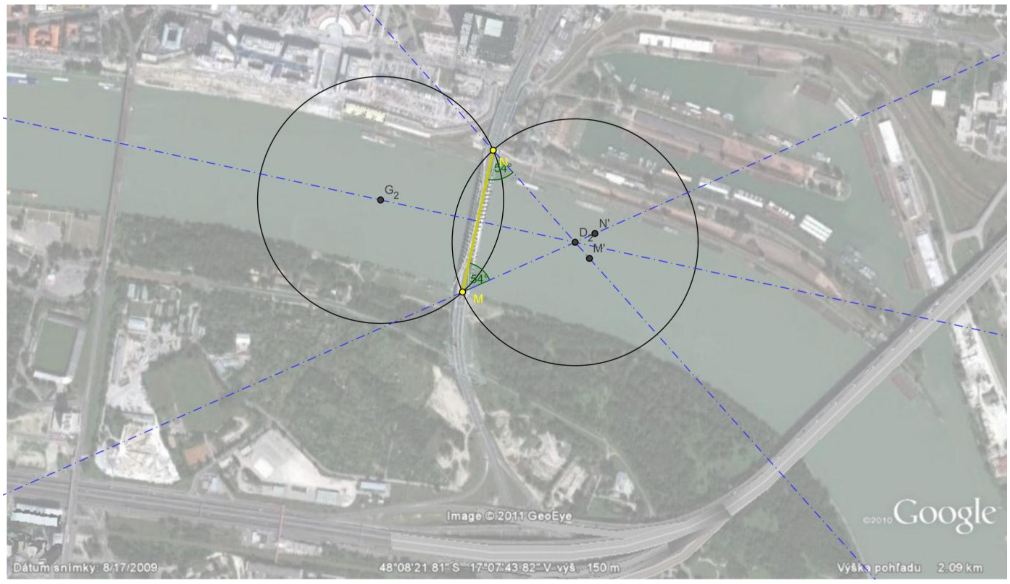



**Fig. 22** Solution of initial problem (Google maps)

**Fig. 23** Apollo Bridge (wiki media)

## 6 Further investigation

How does the situation change when we use camera with different focus instead?

How ca we find a place on the promenade from which we can see two bridges under the best angle? How does the situation change if the promenade won't be perpendicular to the bridge in problem 2? What will change if the promenade will be curved as it is between the Bridges Apollo and the Harbour Bridge? So where exactly is the best spot to look at the Apollo Bridge?

These problems can lead to the Apollonius problems. Maybe it is only a coincidence that the Apollo Bridge let us discover Apollonius problem. However, in reality there are many other examples that can be used not only in math class.

It is also important to let the students rewrite their discoveries in consistent economical manner that will be usable for further investigation and problem solving.

## 7 Discussion

In our investigation we used a lot of approximation by sketching into a Goggle Map in Geogebra. These results are only informative, but they satisfy our purposes for development of mathematical concept. It is appropriate to discuss this issue with students. To extend our accuracywe can use GPS device and introduce coordinates that will help us identify "the best spot" .

There are many ways which circles are studied and used in. In our article we used circles as the concept known to students. Within the inquiry circles helps them to understand new concept through solving a problem. Once more time we would like to stress the importance of investigation with circles. Of course, for this article the topic is too wide. We wanted to show at least a little example how to approach this concept and give the reader a starting point for further investigation.

## 8 Conclusion

The process of investigation and inquiry is the essence of mathematics, where people try to answer questions based on their inquiry. We would like to simulate this process that could be used in our classrooms.

Unsolved problems are an engine that activate mathematicians' curiosity and cause the desire to find the answer. So do us. We understand the unsolved problem as an engine that causes "need" of students to find the solution. That is why we wanted partly reverse traditional textbook tasks back to the nature of mathematics investigation.

Of course we don't deny the advantage of theory building and deductive thinking. We understand this process of finding own path as a dynamical process in mathematics education that can supports students' conceptual understanding.

## References

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