Lifelong Learning Programme

DynaMAT

How to add infinitely looong sums ...

Michaela Klepancová, Marek Varga, Lucia Záhumenská

Addition belongs to basic arithmetic operations. This well known and relatively simple operation in case of adding two, three, four, etc. numbers becomes an unexpected problem in case of addition of infinite number of numbers (addends). This problem was for the first time encountered in the ancient Greece.

Let us mention two stories, known as Zeno's paradoxes. In the first one, let us imagine an archer in a d distance from the target he is aiming at. An arrow thus has to pass the whole distance d before it reaches the target. It also means that an arrow has to pass the half of the distance, i.e. the distance $\frac{d}{2}$. Then it has to move through the half of the rest of the distance, which is $\frac{d}{4}$. From the remaining rest of the distance it has to pass at first again just the half, that is the distance $\frac{d}{8}$, etc. It seems as if the loosened arrow has to pass an exact remaining distance to its target all the time, in other words – it never reaches its target.

Similar point can be found in the story about Achilles and the tortoise. Achilles, one of the biggest warriors of Greek mythology, is in a footrace with the tortoise. His self-esteem allows the tortoise a head start, let us label this distance α. After the start, Achilles runs, of course, faster, and without any problems passes the distance α. However, the tortoise did not wait for him and meanwhile moved on the distance β. It will then take Achilles some further period of time to run that distance, let it be arbitrarily short, in which period the tortoise will advance another tiny distance γ farther. Thus, he has to reach this third point, but whenever Achilles reaches somewhere the tortoise has been, he still has farther to go, because the tortoise moves ahead. We can observe that even a thoroughly chosen competitor can never overtake even the slowest tortoise.

So far the verbal analysis of these situations. However, reality and experience teach us that there would hardly be anyone facing the shooting – as a matter of fact that the bullet never reaches him/her. And maybe even less people would race with the tortoise; they are convinced that they can overreach it easily. Why then such an apparent contradiction appeared in the analysis of the stated situations?

The origin of our problems (or rather the problems of Classical Greek mathematicians and philosophers) probably lies in the fact that the motion of an arrow – a dynamic action – was split into few static situations. Nevertheless, how many such static situations shall we take to "put together" the whole dynamic process? After transforming this question into mathematics we realize that we need to add infinitely many numbers.

Human imagination in the mentioned times (and surely even today) refused to accept the fact that by adding infinitely many numbers (or more precisely, infinitely many positive numbers) a finite product – a finite real number can be obtained. In order to illustrate the above mentioned fact, we will try to bring it closer using simple figures.

Construct a square given a side $a = 1$. Divide the square into two identical rectangles, their areas are obviously equal $\frac{1}{2}$. Choose one of them and divide it again to two identical parts – this time there are identical squares with the area $\frac{1}{4}$. Pick up one of them and divide it into two identical rectangles with identical areas $\frac{1}{8}$. Infinitely continuing in this process, the whole basic square with the area $S = 1$ will be filled (see Figure 1 a, b, c, d, e, f). Natural conclusion is thus the statement that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}+...=1.$ *

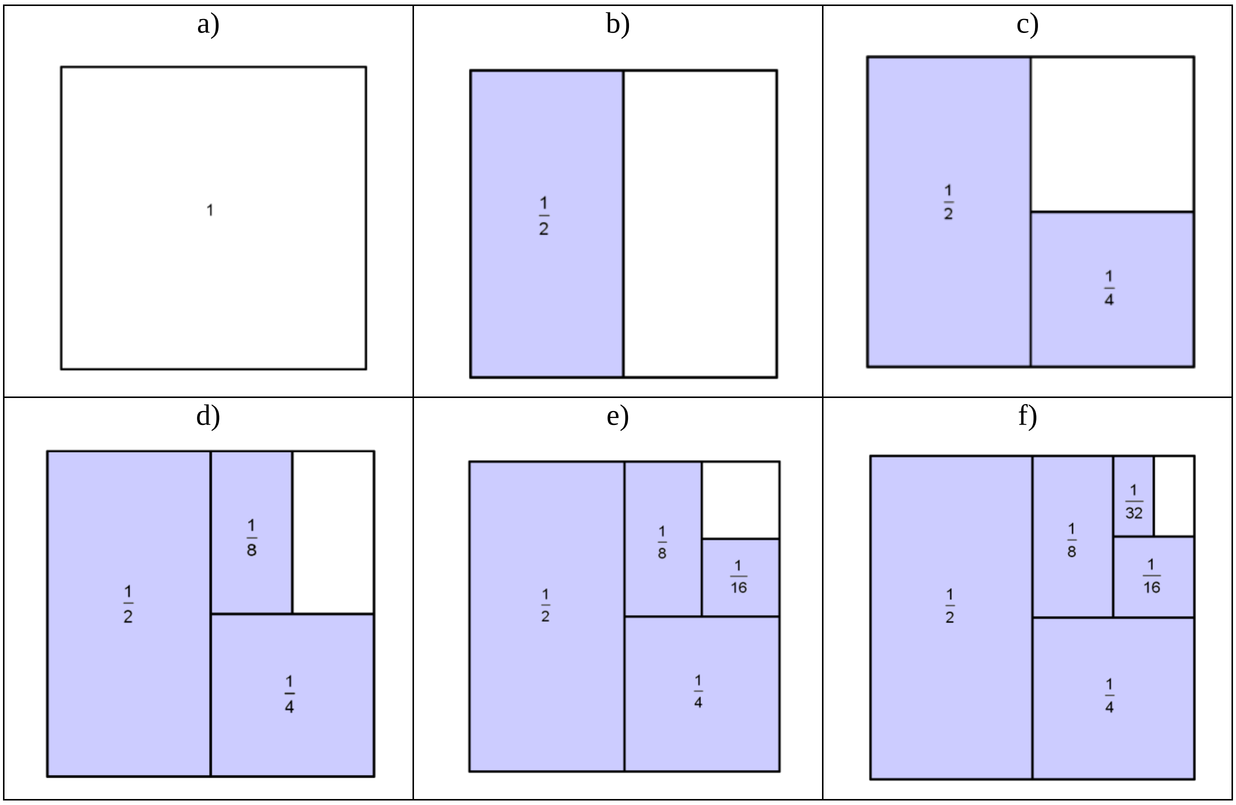

Fig. 1 Numerical series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}+...$

Let us mention that, as usually in mathematics, there is quite convenient symbolism also in case of infinite numerical series representation, i.e. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}+...=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$.

Let us try if the above stated result is obtained independently of the way how the given square is divided. For that purpose, construct again a square with a side $a=1$. Divide it by a diagonal into two parts – two right triangles with equal areas $\frac{1}{2}$. Choose one of them and divide it by an altitude onto the hypotenuse into two other identical right triangles with the areas $\frac{1}{4}$. Divide one of them with an altitude onto the hypotenuse into another right triangles, this time their areas equal $\frac{1}{8}$. If continuing in this process to infinity (see Figures 2 a, b, c, d), by union of all the obtained triangles we gain the original square, therefore by adding areas of all the obtained right triangles we get the area of the given square. Thus, it can again be concluded that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}+...=1$.

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* If we look similarly at the case of a flying arrow, it can easily be proved that an arrow reaches the target in the distance d from the archer. For the sum of the sections, which an arrow crosses, applies $\frac{d}{2}+\frac{d}{4}+...+\frac{d}{2^n}+...=$ $=d\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^n}+...\right)=d$.

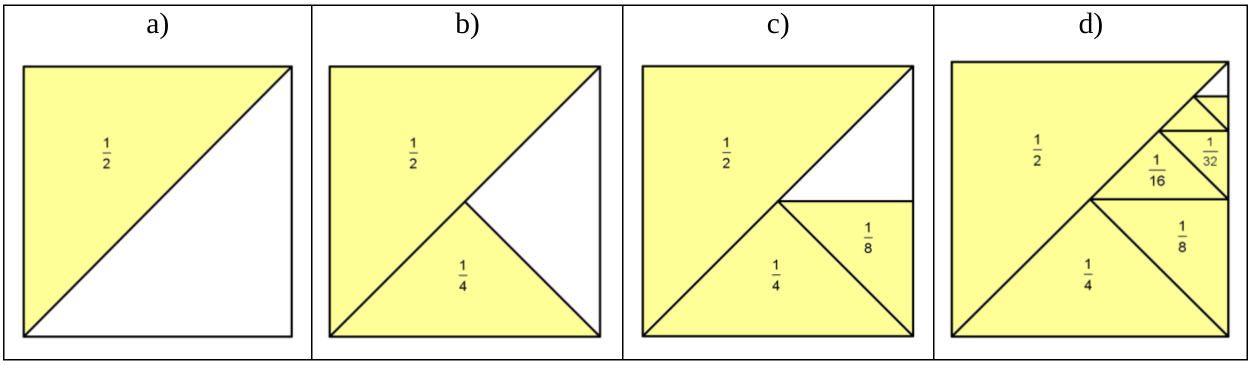


Fig. 2 Numerical series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}+...$

With slight changes these ideas could be transferred also into space, if we consider a cube with an edge $a=1$. We can easily find two divisions of a cube equivalent to described divisions of a square, so that the volumes of obtained solids equal the volume of the whole cube. We would thus confirm that $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n=1$. Let us add – this time without the detailed analysis – one more view of the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}+...$. A square with a side $a=1$ would be again a help. Find the midpoints of its sides. By their matching we obtain another square, and we continue in this process. Then it is from the Figure 3 clear that $\frac{1}{8}+\frac{1}{16}+\frac{1}{32}...+\frac{1}{2^{n+2}}+...=\frac{1}{4}$. By adding the first two terms of the series we obtain the equality $\frac{1}{2}+\frac{1}{4}+\left(\frac{1}{8}+\frac{1}{16}+\frac{1}{32}...+\frac{1}{2^{n+2}}+...\right)=\frac{3}{4}+\frac{1}{4}=1$.

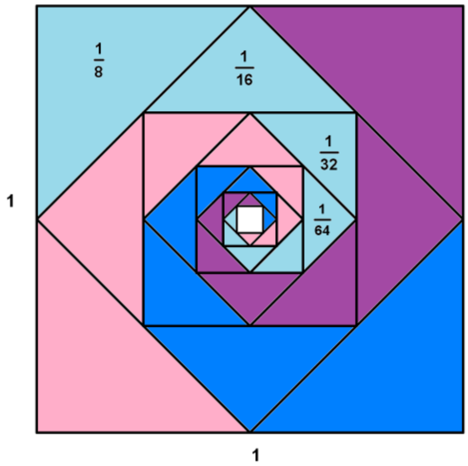


Fig. 3 Numerical series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}+...$

Stated ideas suggest that the sum of the infinitely many positive addends really can be a finite real number. Let us prove that series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}+...$ does not have any "supreme" position – that it can be arrived at the same conclusion when considering many other infinite numerical series. We will use simple illustrations for better demonstration.

A square proved to be a very good device in the above example situations – construct therefore once again a square given a side $a=1$. Divide it with two midsegments into four smaller squares, their areas equal $\frac{1}{4}$. Choose one of them and divide it in the same way into four identical smaller squares, their areas being $\frac{1}{16}$. As can be expected, again pick up one of these squares and divide it in the stated way into identical squares with areas equal $\frac{1}{64}$. Repeat the division of a shape infinitely many times (see Figures 4 a, b, c, d, e, f), and observe only those squares that are colored in the figure. The sum of all their areas, i.e. the sum $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots+\frac{1}{4^n}+\ldots$, obviously substitutes one third of the whole area of the original square. Thus we get $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots+\frac{1}{4^n}+\ldots=\frac{1}{3}$, i.e. $\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n=\frac{1}{3}$.

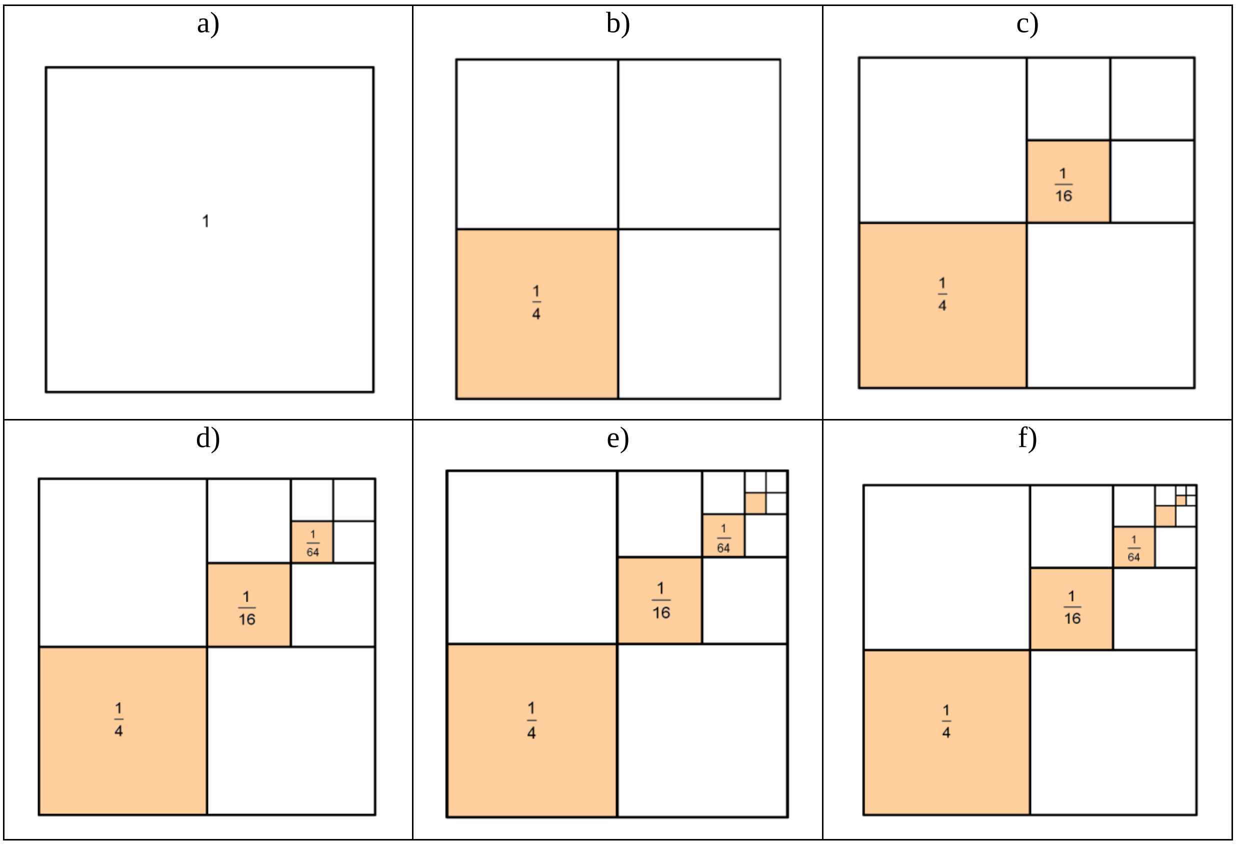


Fig. 4 Numerical series $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots+\frac{1}{4^n}+\ldots$

Once more, do not trust everything you see, and let us try to prove the obtained result in a different way. We will stop referring to squares exclusively and try to use equilateral triangle this time (the length of the side does not make any difference), let its area be labeled with letter *S*. By matching its midpoints we obtain four identical equilateral triangles, areas of which obviously equal $\frac{S}{4}$. Choose one of them and with the help of midsegments create four identical equilateral triangles, their areas are $\frac{S}{16}$. After the third analogical step we get another quaternion of equilateral triangles, this time their areas being $\frac{S}{64}$. We can continue in this process infinitely (see Figures 5 a, b, c), gaining thus the set of triangles highlighted in color in our figure. Considering the area they comprise in the original triangle, it can be claimed that the equation $\frac{S}{4}+\frac{S}{16}+\frac{S}{64}+\ldots+\frac{S}{4^n}+\ldots=\frac{S}{3}$ is valid, resp. after the simplification by a nonzero number *S* we gain equivalent equation $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots+\frac{1}{4^n}+\ldots=\frac{1}{3}$.

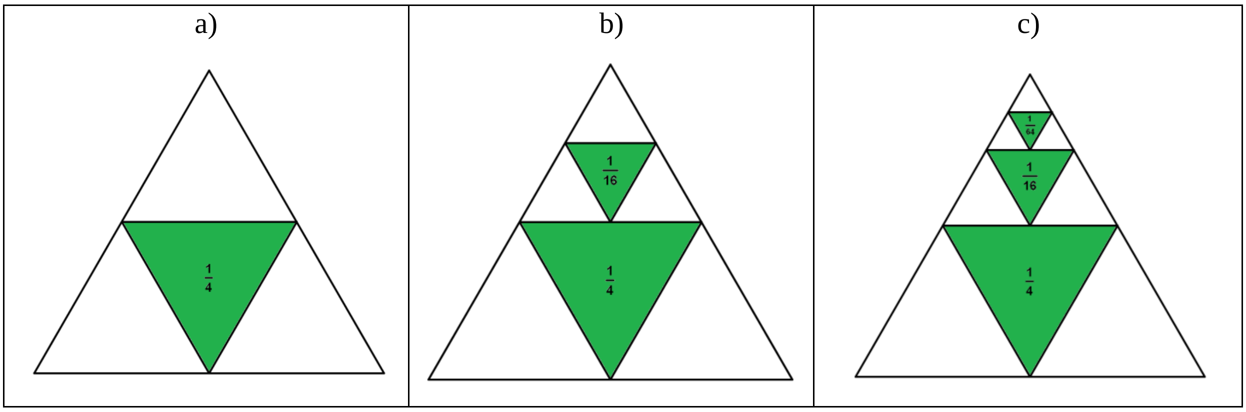
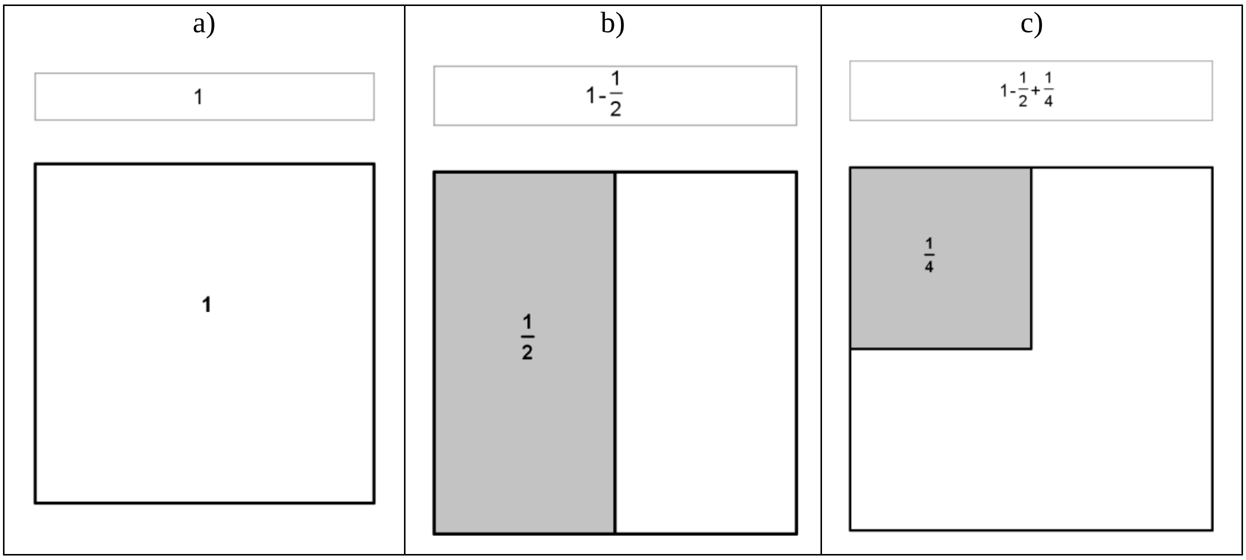


Fig. 5 Numerical series $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots+\frac{1}{4^n}+\ldots$

Up to now, the emphasis was put – within highlighting the paradox that the sum of the infinitely many addends is a finite number – only on the sum of positive numbers. Further, let us think about more general problem: Can the sum of infinitely many positive and negative addends be a finite number? Let us approach this problem from the geometrical point of view, too.

At the beginning, let us choose the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\left(-\frac{1}{2}\right)^{n-1}+\ldots$ as we can make use of one of the previously listed figures while exploring this problem. We construct a square with a side $a=1$ again and divide it in the way described in the figure 4. Despite that, this view is going to be different – while in the previous situation the colored shapes were observed, in this case the rest of the figure is going to be interesting for us. The description of this situation is included directly within the figures 6 a, b, c, d, e, f, g, h, i. It should be once again emphasized that this time the off-color part of the illustration is important. They after all build up the two-thirds part of the original square and consequently we can write $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\left(-\frac{1}{2}\right)^{n-1}+\ldots=\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^{n-1}=\frac{2}{3}$.



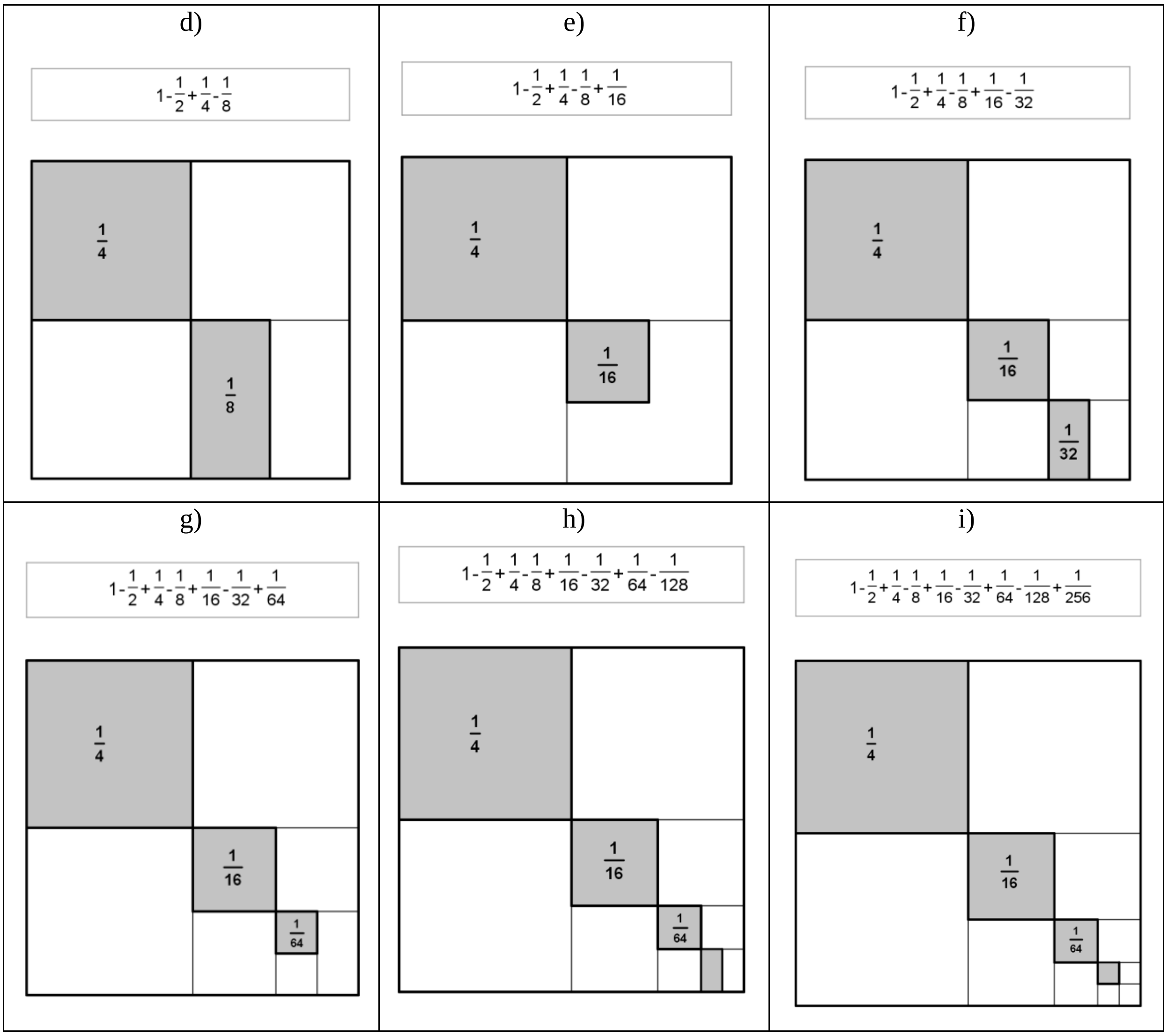


Fig. 6 Numerical series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\left(-\frac{1}{2}\right)^{n-1}+\ldots$

Let us add at least one similar sample, let us deal with the infinite numerical series $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{12}+\ldots+\left(-\frac{1}{3}\right)^{n-1}+\ldots$. Although it is hard to believe, it is enough to use a square with the side $a=1$ again to prove this situation. The division principle is understandable from the series of figures 7 a, b, c, d, e, f. It should be added that the attention should be paid to the off-color shapes obtained in the process. It is not hard to see that the infinite repetition of this process leads towards gaining the shape, the area of which equals three quarters of the whole square. That means that we obtain the equation $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{12}+\ldots+\left(-\frac{1}{3}\right)^{n-1}+\ldots=\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^{n-1}=\frac{3}{4}$.

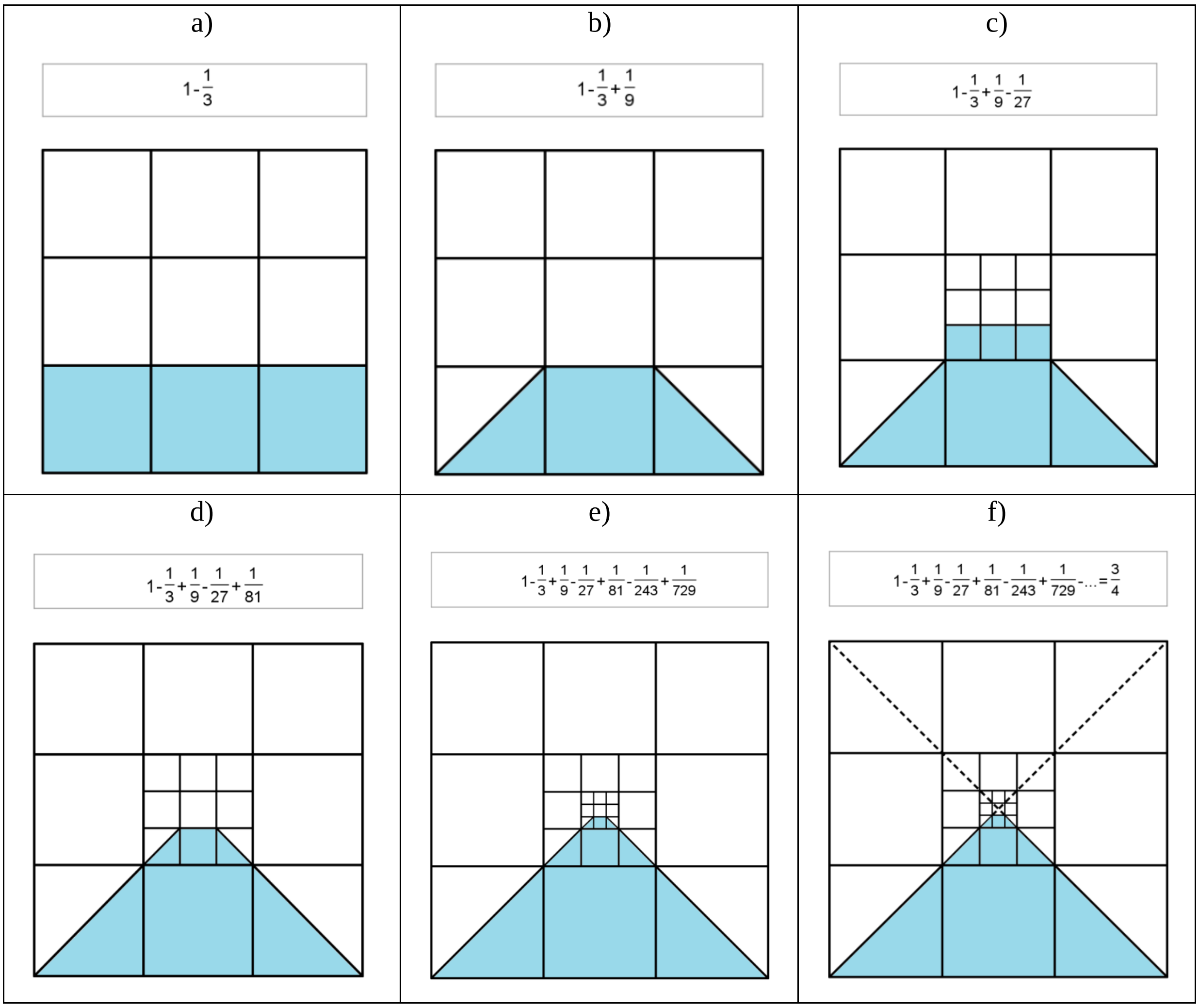


Fig. 7 Numerical series $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{12}+\dots+\left(-\frac{1}{3}\right)^{n-1}+\dots$

Infinite numerical series which we dealt with can be generally recorded as $q+q^2+q^3+\dots+q^n+\dots$ resp. $\sum_{n=1}^{\infty} q^n$, and they are referred to as geometric series, number q is the constant multiple of the series. It should be mentioned that values of constant multiple $q\in(-1;1)$** were not chosen by chance, as these are the cases when the sum of geometric series is finite. If the symbol s refers to the sum of series $\sum_{n=1}^{\infty} q^n$, then we have already encountered the following situations: $q=\frac{1}{2} \Rightarrow s=1$, $q=\frac{1}{4} \Rightarrow s=\frac{1}{3}$, $q=-\frac{1}{2} \Rightarrow s=\frac{2}{3}$, $q=-\frac{1}{3} \Rightarrow s=\frac{3}{4}$. Unfortunately, it is possible that the sum of the geometric series is not dependant only on its constant multiple. Therefore, instead of the hypotheses about the form of the formula for the sum s, let us try to discover the demanded formula with the help of geometrical sketches and reflections.

** We also do not deal with the trivial case of $q=0$.

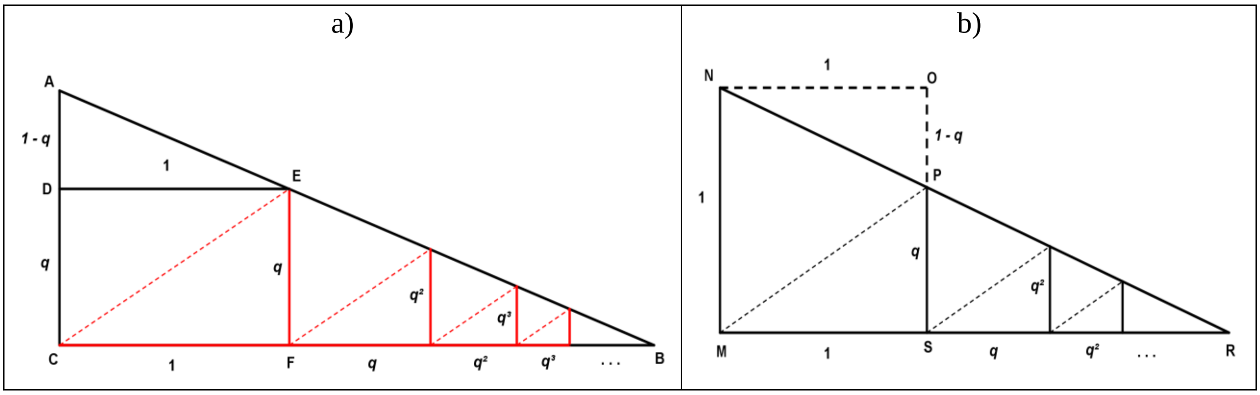


Fig. 8 Numerical series $1+q+q^2+...+q^n+...$

At first, construct the rectangle CFED, whereby $|CF|=1$, $|CD|=q$. Construction of a point A is clear from the figure 8a, as well as point B emerges as an intersection of half-lines AE and CF. The similarity of the triangle CFE and all other colored triangles implies that the lengths of given horizontal sections equal q, q^2, q^3, etc. From the similarity of triangles DEA and CBA we obtain the equation $1+q+q^2+...+q^n+...=\dfrac{1}{1-q}$.

It should also be noted that this result was gained thanks to the "completion" of a square MSON in the figure 8b and through the use of similarity of triangles NPO and MRN.

Thus a simple formula for the sum of the series $\sum_{n=0}^{\infty} q^n$ was acquired. We admit that the situation was simplified thanks to the first term in the series, which is number 1. How it would like in the general formula, i.e. in case of series $a+aq+aq^2+...+aq^n+...$?

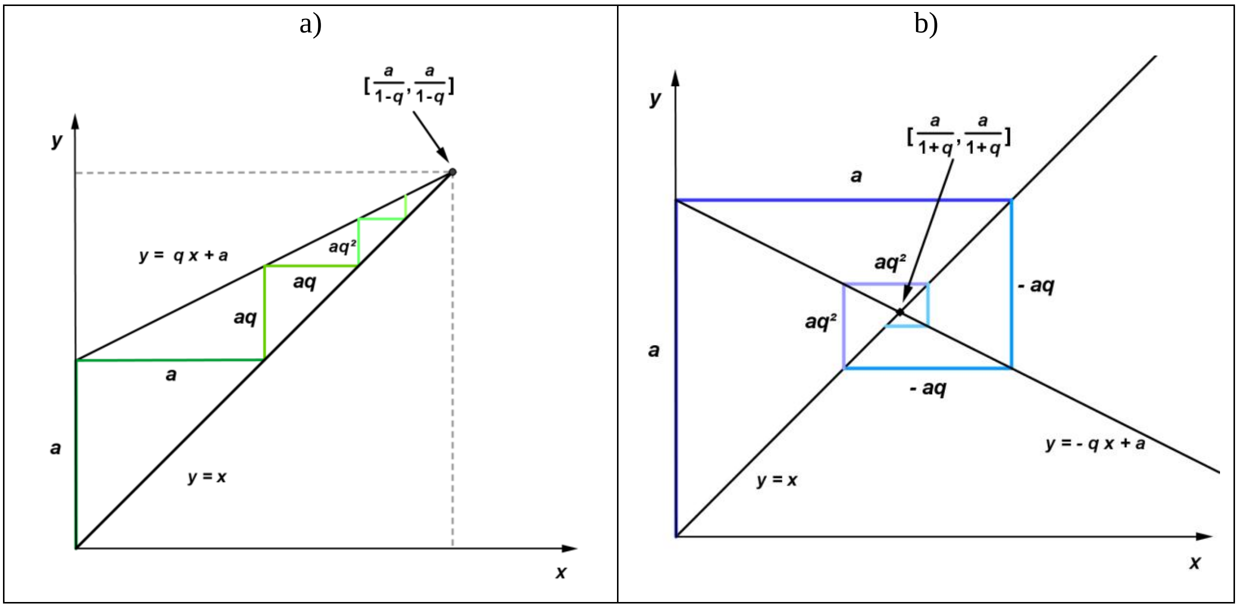


Fig. 9 Numerical series $a+aq+aq^2+...+aq^n+...$, resp. $a-aq+aq^2-...+a(-q)^{n-1}+...$

Observe figure 9a. Construct straight lines $y=x$ and $y=qx+a$ in the system of coordinates. The second line intersects an axis o_y in the a distance from the origin of coordinates. The length of the segment led from the intersection parallel to the axis o_x up to the intersection with the straight line

$y = x$ equals also *a*, it is the axis of a quadrant. Length of another vertical segment is *aq*, it is the value of a function $y = qx + a$ in the point *a*. Further procedure of the construction of colored parts is obvious. However, both coordinates of the point of intersection of the given lines correspond the sum of the individual sections, wherefrom the equation $a + aq + aq^2 + ... + aq^n + ... = \frac{a}{1-q}$.

Analogically, in figure 9b the addition of terms of series $a - aq + aq^2 - ... + a(-q)^{n-1} + ...$ using lines $y = x$ and $y = -qx + a$ is illustrated.

It appears that the sum of infinite geometric series is dependant on the first term of the series as well as on its constant variable. Finally, let us verify the obtained formula by a simple calculation. Let us assume that the sum of geometric series $\sum_{n=0}^{\infty} aq^n$ equals value *s*, i.e. $\sum_{n=0}^{\infty} aq^n = s$. That means that the equation $s = a + aq + aq^2 + ... + aq^n + ...$ is valid. Multiply the equation by a nonzero number *q*, so that we get $sq = aq + aq^2 + aq^3 + ... + aq^{n+1} + ...$ By subtracting these equalities we obtain $s - sq = a$, wherefrom $s = \frac{a}{1-q}$, which is the formula derived from the figures.

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