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How to add infinitely looong sums ... Modelling optical lenses with Dynamic Geometry Software Andreas Ulovec

Michaela Klepancová, Marek Varga, Lucia Záhumenská **1 Introduction**

Addition belongs to basic arithmetic operations. This well known and relatively simple operation in case of adding two, three, four, etc. numbers becomes an unexpected problem in case of addition of ease of dating two, three, four, etc. hambers occomes an analyzed problem in case of datation of infinite number of numbers (addends). This problem was for the first time encountered in the ancient Greece. of making making light visible. This problem was for the first three encountered in the aner glass lenses \mathcal{L} not always available, and adjustments to the system can usually only be set on \mathcal{L}

Let us mention two stories, known as Zeno's paradoxes. In the first one, let us imagine an archer in a *d* distance from the target he is aiming at. An arrow thus has to pass the whole distance d before it

reaches the target. It also means that an arrow has to pass the half of the distance, i.e. the distance 2 $\frac{d}{\cdot}$. situation before the change and after the change – but it is not exactly a gradual change that lets them eaches the target. It also means that an arrow has to pass the half of the distance, i.e. the distance

Then it has to move through the half of the rest of the distance, which is 4 $\frac{d}{dx}$. From the remaining rest Then it has to move through the hair of the rest of the distance, which is $\frac{1}{4}$. From the remaining i reflection and refraction – not *instead* of the actual experiment (if one sees experiments only in

of the distance it has to pass at first again just the half, that is the distance 8 $\frac{d}{dx}$, etc. It seems as if the of the distance it has to pass at first again just the half, that is the distance $\frac{u}{r}$, etc. It seems as if

loosened arrow has to pass an exact remaining distance to its target all the time, in other words – it never reaches its target. for mathematics teachers. We have in the mathematics? The mathematics? There is a lot of it in the mathematics? If a ray of it is a ray of it is a ray of it in the mathematics? The mathematics? If a ray of it is a ray of

Similar point can be found in the story about Achilles and the tortoise. Achilles, one of the biggest which the light is reflected in the sterp about Fermines and the tortoise. Fremines, one of the enggest warriors of Greek mythology, is in a footrace with the tortoise. His self-esteem allows the tortoise a warriors of Greek hrydnology, is in a footdace with the tortorse. This sen-esteem anows the tortorse a
head start, let us label this distance α . After the start, Achilles runs, of course, faster, and without any problems passes the distance α. However, the tortoise did not wait for him and meanwhile moved on μ become passes the distance d. Trowever, the tortoise did not want for fifth and meanwhile moved on the distance, let it be the distance, let it be arbitrarily short, in which period the tortoise will advance another tiny distance γ farther. Thus, he has to reach this third point, but whenever Achilles reaches somewhere the tortoise has been, he still has farther to go, because the tortoise moves ahead. We can observe that even a thoroughly chosen commetities can pay a pure surface a proper surface competitor can never overtake even the slowest tortoise. near the centre of the centre of the centre of the calculation being more of the calculation of the calculations of $\frac{1}{2}$ Ω_{g} is a lens with the properties of a length and without a lens with a lens with Ω_{g} let Ω_{g}

So far the verbal analysis of these situations. However, reality and experience teach us that there would hardly be anyone facing the shooting – as a matter of fact that the bullet never reaches him/her. And maybe even less people would race with the tortoise; they are convinced that they can overreach it easily. Why then such an apparent contradiction appeared in the analysis of the stated situations?

The origin of our problems (or rather the problems of Classical Greek mathematicians and philosophers) probably lies in the fact that the motion of an arrow $-$ a dynamic action $-$ was split into few static situations. Nevertheless, how many such static situations shall we take to "put together" the whole dynamic process? After transforming this question into mathematics we realize that we need to add infinitely many numbers.

Human imagination in the mentioned times (and surely even today) refused to accept the fact that by adding infinitely many numbers (or more precisely, infinitely many positive numbers) a finite product – a finite real number can be obtained. In order to illustrate the above mentioned fact, we will try to bring it closer using simple figures.

Construct a square given a side $a = 1$. Divide the square into two identical rectangles, their areas are obviously equal $\frac{1}{2}$ 2 . Choose one of them and divide it again to two identical parts – this time there are identical squares with the area $\frac{1}{x}$ 4 . Pick up one of them and divide it into two identical rectangles with identical areas $\frac{1}{2}$ 8 . Infinitely continuing in this process, the whole basic square with the area $S = 1$ will $\frac{1}{\sqrt{2}}$ **Fig.1** Reflection of light at a plane surface.

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be filled (see Figure 1 a, b, c, d, e, f). Natural conclusion is thus the statement 1 1 1 1 ¹ ¹ that $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + ... + \frac{1}{2^n} + ... = 1$ $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = 1.$ $2\quad 4\quad 8$

Fig. 1 Numerical series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{8} + ...$ **Fig. 1** Numerical series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} + ...$

Let us mention that, as usually in mathematics, there is quite convenient symbolism also in case of infinite numerical series representation, i.e. 1 $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = \sum_{n=1}^{\infty} \left(\frac{1}{n} \right)$ $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = \sum_{n=1}^{\infty} \left(\frac{1}{2} \right)$ *n n n* ∞ = of the member dial, as assains in maintenances, there is quite convenient symmetric symmetr $\frac{2}{\pi}$ **r** $\frac{4}{\pi}$ **o** $\frac{2}{\pi}$ $\frac{n=1}{2}$

Let us try if the above stated result is obtained independently of the way how the given square is divided. For that purpose, construct again a square with a side $a = 1$. Divide it by a diagonal into two parts – two right triangles with equal areas $\frac{1}{2}$ 2 . Choose one of them and divide it by an altitude onto the hypotenuse into two other identical right triangles with the areas $\frac{1}{1}$ 4 . Divide one of them with an altitude onto the hypotenuse into another right triangles, this time their areas equal $\frac{1}{\epsilon}$ 8 . If continuing in this process to infinity (see Figures 2 a, b, c, d), by union of all the obtained triangles we gain the original square, therefore by adding areas of all the obtained right triangles we get the area of the given square. Thus, it can again be concluded that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n} + ... = 1$ $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = 1$.

$$
= d\left(\frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} + \dots\right) = d.
$$

 * If we look similarly at the case of a flying arrow, it can easily be proved that an arrow reaches the target in the distance *d* from the archer. For the sum of the sections, which an arrow crosses, applies $\frac{a}{2} + \frac{a}{4} + ... + \frac{a}{2^n} + ...$ the sum of the sections, which an arrow crosses, applies $\frac{d}{2} + \frac{d}{2} + ... + \frac{d}{2} + ...$ This project has been funded with support from the European Commission in its Lifelong Learning Programme \mathcal{L}

With slight changes these ideas could be transferred also into space, if we consider a cube with an whilf slight changes these fields could be transferred also like space, if we consider a cube with an edge $a = 1$. We can easily find two divisions of a cube equivalent to described divisions of a square, so that the volumes of obtained solids equal the volume of the whole cube. We would thus confirm that 1 $\left(\frac{1}{2}\right)^n = 1$ 2 *n n* ∞ = $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$. Let us add – this time without the detailed analysis – one more view of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2^n} + \dots$ $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots$. A square with a side $a = 1$ would be again a help. Find the midpoints of its sides. By their matching we obtain another square, and we continue in this process. Then it is from the Figure 3 clear that $\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{2^{n+2}}$ $\frac{1}{8} + \frac{1}{16} + \frac{1}{23} + \frac{1}{24} + \dots = \frac{1}{4}$ Figure 3 clear that $\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots + \frac{1}{2^{n+2}} + \dots = \frac{1}{4}$. By adding the first two terms of the series we obtain the equality $\frac{1}{2} + \frac{1}{4} + \left(\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{2^{n+2}} + \dots\right) = \frac{3}{4} + \frac{1}{4} = 1$ $\frac{1}{2} + \frac{1}{4} + \left(\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots + \frac{1}{2^{n+2}} + \dots\right) = \frac{3}{4} + \frac{1}{4}$ $8^{+}16^{+}32^{...+}2^{n+2+...-4}$. *by adding the*
+ $\frac{1}{4} + \left(\frac{1}{8} + \frac{1}{16} + \frac{1}{32}... + \frac{1}{2^{n+2}} + ...\right) = \frac{3}{4} + \frac{1}{4} = 1.$. light $u = 1$. We can easily thin two divisions of a cube equivalent to d $T \left(1 \right)^n$ hat $\sum_{n=1}^{\infty}$ and *i* and *i* the set in the set the set of the set the set of the set of the set of the set of the set $\frac{1}{n-1}(2)$ for mathematics teachers. We have in the mathematics? The mathematics? There is a lot of it in the mathematics? The mat $\frac{1}{k}$ + $\frac{1}{k}$ + $\frac{1}{k}$ + \ldots + $\frac{1}{k}$ + \ldots A square with a side $a = 1$ would be again a help. Find the midpoints of 2 4 8 $2ⁿ$ igure 5 clear that $\frac{1}{8} + \frac{1}{16} + \frac{2}{32} + \cdots + \frac{1}{2^{n+2}} + \cdots - \frac{1}{4}$. By adding the first two terms of the series we obt $\frac{1}{\sqrt{1-\frac{1$ be equality $\frac{1}{t} + \frac{1}{t} = 1$. $\frac{2}{2}$ 4 (8 16 32 $\frac{2^{n+2}}{1}$) 4 4

Fig. 3 Numerical series $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + ... + \frac{1}{6} + ...$ $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots$

Stated ideas suggest that the sum of the infinitely many positive addends really can be a finite real number. Let us prove that series $\frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots$ $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots$ does not have any "supreme" position – that it can be arrived at the same conclusion when considering many other infinite numerical series. We will use simple illustrations for better demonstration.

A square proved to be a very good device in the above example situations – construct therefore once A square proved to be a very good device in the above example studions $\frac{1}{2}$ construct therefore once again a square given a side $a = 1$. Divide it with two midsegments into four smaller squares, their areas equal $\frac{1}{1}$ 4 reas equal $\frac{1}{x}$. Choose one of them and divide it in the same way into four identical smaller squares, $\frac{1}{4}$. Choose one of them and dryne it in the same way mo four nemicar smaller squ

their areas being $\frac{1}{1}$ 16 heir areas being $\frac{1}{16}$. As can be expected, again pick up one of these squares and divide it in the stated way into identical squares with areas equal $\frac{1}{x}$ 64 way into identical squares with areas equal $\frac{1}{n}$. Repeat the division of a shape infinitely many times (see Figures 4 a, b, c, d, e, f), and observe only those squares that are colored in the figure. The sum of all their areas, i.e. the sum $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + ... + \frac{1}{16} + ...$ Ill their areas, i.e. the sum $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots + \frac{1}{4^n} + \dots$, obviously substitutes one third of the whole area $4\quad 16\quad 64\quad 4^n$

of the original square. Thus we get $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + ... + \frac{1}{4^n} + ... = \frac{1}{2}$ $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots + \frac{1}{4^n} + \dots = \frac{1}{3}$, i.e. 1 1 ⁿ 1 4) 3 *n n* ∞ \equiv of the original square. Thus we get $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots + \frac{1}{4^n} + \dots = \frac{1}{3}$, i.e. $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{3}$. of the original square. Thus we get $\frac{1}{4} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} = \frac{1}{2}$, i.e. $\sum |\frac{1}{4}| = \frac{1}{2}$. $4\overline{10}$ 04 $\overline{4}$ 0 $n_{\text{H}} = \frac{1}{4}$

Once more, do not trust everything you see, and let us try to prove the obtained result in a different way. We will stop referring to squares exclusively and try to use equilateral triangle this time (the length of the side does not make any difference), let its area be labeled with letter *S*. By matching its midpoints we obtain four identical equilateral triangles, areas of which obviously equal 4 $\frac{S}{\cdot}$. Choose one of them and with the help of midsegments create four identical equilateral triangles, their areas are 16 $\frac{S}{\sqrt{S}}$. After the third analogical step we get another quaternion of equilateral triangles, this time their areas being 64 $\frac{S}{S}$. We can continue in this process infinitely (see Figures 5 a, b, c), gaining thus the set of triangles highlighted in color in our figure. Considering the area they comprise in the original triangle, it can be claimed that the equation $\frac{S}{A} + \frac{S}{A} + \frac{S}{A} + \dots + \frac{S}{A^n} + \dots$ $\frac{5}{4} + \frac{5}{16} + \frac{5}{64} + \dots + \frac{5}{4^n} + \dots = \frac{5}{3}$ boot in our rigure. Considering the area they comprise in the original
hat the equation $\frac{S}{4} + \frac{S}{16} + \frac{S}{64} + ... + \frac{S}{4^n} + ... = \frac{S}{2}$ is valid, resp. after the simplification by a nonzero number *S* we gain equivalent equation $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + ... + \frac{1}{4^n} + ... = \frac{1}{2}$ implification by a nonzero number S we gain equivalent equation $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + ... + \frac{1}{4^n} + ... = \frac{1}{3}$. Improvements by a nonzero namely s we gain equivariant equation $4 \times 16 \times 64 \times 4^n \times 3$

Up to now, the emphasis was put $-$ within highlighting the paradox that the sum of the infinitely many addends is a finite number – only on the sum of positive numbers. Further, let us think about more addends is a finite number – only on the sum of positive numbers. Further, let us think about more general problem: Can the sum of infinitely many positive and negative addends be a finite number? Let us approach this problem from the geometrical point of view, too. reflection and refraction – not *instead* of the actual experiment (if one sees experiments only in

At the beginning, let us choose the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots + \left(-\frac{1}{8} \right)^{n-1} + \dots$ $\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots + \left(-\frac{1}{2}\right)$ $-\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots + \left(-\frac{1}{2}\right)^{n-1} + \dots$ as we can make use of for mathematics teachers. We have in the mathematics? The mathematics? There is a lot of it is a lot of it in the mathematics? The mathematics? The mathematics? The mathematics? The mathematics? The mathematics? If a ray At the beginning, let us choose the series $1-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}+\ldots+\frac{1}{n-1}$ +... as we can make use and continues the glass and continues the glass and continues the same happens when the s

one of the previously listed figures while exploring this problem. We construct a square with a side $a = 1$ again and divide it in the way described in the figure 4. Despite that, this view is going to be d ifferent – while in the previous situation the colored shapes were observed, in this case the rest of the figure is going to be interesting for us. The description of this situation is included directly within the figures 6 a, b, c, d, e, f, g, h, i. It should be once again emphasized that this time the off-color part of \mathcal{L}_{G} is the properties of a lens with \mathcal{L}_{G} is important. They after all huild un the two-thirds part of the original square is

the illustration is important. They after all build up the two-thirds part of the original square and
consequently we can write
$$
1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ... + \left(-\frac{1}{2}\right)^{n-1} + ... = \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1} = \frac{2}{3}
$$
.

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Fig. 6 Numerical series $\frac{1}{1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\left(-\frac{1}{2}\right)^{n-1}+\ldots}$ $\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots + \left(-\frac{1}{2}\right)$ **Fig. 6** Numerical series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + ... + \left(-\frac{1}{2}\right)^{n-1} + ...$ **the angle of** $\frac{1}{1-\frac{1}{n}} + \frac{1}{n-\frac{1}{n}} + \frac{1}{n-\frac{1}{n}}$

Let us add at least one similar sample, let us deal with the infinite numerical series $1 - \frac{1}{2} + \frac{1}{9} - \frac{1}{12} + ... + \left(-\frac{1}{2}\right)^{n-1} + ...$ $\frac{1}{3} + \frac{1}{9} - \frac{1}{12} + ... + \left(-\frac{1}{3}\right)$ $-\frac{1}{3} + \frac{1}{9} - \frac{1}{12} + ... + \left(-\frac{1}{3}\right)^{n-1} + ...$. Although it is hard to believe, it is enough to use a square with the side $a = 1$ again to prove this situation. The division principle is understandable from the series of figures 7 a, b, c, d, e, f. It should be added that the attention should be paid to the off-color shapes obtained in the process. It is not hard to see that the infinite repetition of this process leads towards gaining the shape, the area of which equals three quarters of the whole square. That means that we ree quarters of the
 $\frac{1}{1}$ $\frac{\infty}{1}$ $\left(\frac{1}{1}\right)^{n-1}$ the area of which equals three quarters of the wh
 $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{12} + ... + \left(-\frac{1}{3}\right)^{n-1} + ... = \sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^{n-1} = \frac{3}{4}$ three quarters of t hree quarters of the wh
 $\frac{m}{2}$ (1)ⁿ⁻¹

obtain the equation $\frac{1}{3} + \frac{1}{9} - \frac{1}{12} + \dots + \left(-\frac{1}{3} \right)^{n-1} + \dots = \sum_{n=1}^{\infty} \left(-\frac{1}{3} \right)^{n-1} = \frac{3}{4}$ *n* = e area of which equals three quarters of the whole s
 $-\frac{1}{3} + \frac{1}{9} - \frac{1}{12} + ... + (-\frac{1}{3})^{n-1} + ... = \sum_{n=1}^{\infty} (-\frac{1}{3})^{n-1} = \frac{3}{4}.$

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Fig. 7 Numerical series $\frac{1}{3} + \frac{1}{9} - \frac{1}{12} + ... + \left(-\frac{1}{3}\right)$ the angle of incidence (between the ray of light and the *normal*) is equal to the angle of reflection:

Infinite numerical series which we dealt with can be generally recorded as $q + q^2 + q^3 + ... + q^n + ...$ resp. $\sum q^n$ 1 *n* $\sum_{n=1}^{\infty} q^n$, and they are referred to as geometric series, number *q* is the constant multiple of the $=$ series. It should be mentioned that values of constant multiple $q \in (-1,1)^{**}$ were not chosen by chance, as these are the cases when the sum of geometric series is finite. If the symbol *s* refers to the sum of series $\sum q^n$ 1 *n* ∞ $\sum_{n=1}^{\infty} q^n$, then we have already encountered the following situations: $q = \frac{1}{2} \implies s = 1$ 2 $q=\frac{1}{s} \Rightarrow s=1$, 1 1 4 3 $q = \frac{1}{x} \implies s = \frac{1}{2}, \quad q = -\frac{1}{2} \implies s = \frac{2}{3}$ $2 \begin{array}{c} 2 \end{array}$ 3 $q=-\frac{1}{2} \Rightarrow s=\frac{2}{3}, q=-\frac{1}{2} \Rightarrow s=\frac{3}{4}$ $3 \begin{array}{c} 3 \\ 4 \end{array}$ $q = -\frac{1}{\epsilon} \Rightarrow s = \frac{3}{\epsilon}$. Unfortunately, it is possible that the sum of the geometric series is not dependant only on its constant multiple. Therefore, instead of the hypotheses about the form of the formula for the sum *s*, let us try to discover the demanded formula with the help of geometrical sketches and reflections.

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^{**} We also do not deal with the trivial case of $q = 0$. $\mathcal{L}_{\mathcal{S}}$ -comenius-comeniu

Fig. 8 Numerical series $1+q+q^2+...+q^n+...$ **lig. 8** Numerical series $1+q+q^2+...+q$

At first, construct the rectangle CFED, whereby $|CF| = 1$, $|CD| = q$. Construction of a point A is clear from the figure 8a, as well as point B emerges as an intersection of half-lines AE and CF. The creat from the righte oa, as well as point **B** emerges as an intersection of han-lines AE and CP. The similarity of the triangle CFE and all other colored triangles implies that the lengths of given horizontal sections equal q, q^2 , q^3 , etc. From the similarity of triangles DEA and CBA we obtain the equation $1 + q + q^2 + ... + q^n + ... = \frac{1}{1 -}$ $+q+q^2+...+q^n+...=\frac{1}{1-q}$. $\frac{1}{100}$ mathematics teachers. The mathematic induced in the mathematics in the mathematics of $\frac{1}{2}$ ray of $\frac{$ another part penetrates the glass and continues the same happens when the same happens wh he equation $1+q+q^2+\ldots+q^n+\ldots = \frac{q^2+q^2}{q^2+q^2+1}$. $v_1 - q$

It should also be noted that this result was gained thanks to the "completion" of a square MSON in the figure 8b and through the use of similarity of triangles NPO and MRN. $\mathcal{L}_{\mathcal{D}}$ more complex, and from the equations alone it would be different.

Thus a simple formula for the sum of the series 0 *n n q* ∞ Thus a simple formula for the sum of the series $\sum_{n=0}^{\infty} q^n$ was acquired. We admit that the situation was simplified thanks to the first term in the series, which is number 1. How it would like in the general formula, i.e. in case of series $a + aq + aq^2 + ... + aq^n + ...$? thus a simple formula for the sum of the series $\sum q^2$ was acquired. We admit that the first place.

Fig. 9 Numerical series $a + aq + aq^2 + ... + aq^n + ...$, resp. $a - aq + aq^2 - ... + a(-q)^{n-1} + ...$

Observe figure 9a. Construct straight lines $y = x$ and $y = qx + a$ in the system of coordinates. The second line intersects an axis o_y in the *a* distance from the origin of coordinates. The length of the segment led from the intersection parallel to the axis o_x up to the intersection with the straight line Commission cannot be intersection parameter of the axis v_x up to the intersection with the straig

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 $y = x$ equals also *a*, it is the axis of a quadrant. Length of another vertical segment is *aq*, it is the value $y = x$ equals also *a*, it is the axis of a quadrant. English of abouter vertical segment is *aq*, it is the value of a function $y = qx + a$ in the point *a*. Further procedure of the construction of colored parts is obvious. However, both coordinates of the point of intersection of the given lines correspond the sum

of the individual sections, wherefrom the equation $a + aq + aq^2$... + aq^{n} + ... = $\frac{1}{1}$ $a + aq + aq^2 + ... + aq^n + ... = \frac{a}{1-q}$ of the individual sections, wherefrom the equation $a + aq + aq^2 + ... + aq^n + ... = \frac{a}{1-q}$.

Analogically, in figure 9b the addition of terms of series $a - aq + aq^2 - ... + a(-q)^{n-1} + ...$ using lines $y = x$ and $y = -qx + a$ is illustrated. Analogically, in figure 96 the addition of terms of series $a - aq + aq - ... + a(-q)$ +... using if

It appears that the sum of infinite geometric series is dependant on the first term of the series as well as on its constant variable. Finally, let us verify the obtained formula by a simple calculation. Let us assume that the sum of geometric series 0 *n n aq* ∞ $\sum_{n=0} aq^n$ equals value *s*, i.e. $\mathbf{0}$ *n n* $aqⁿ = s$ ∞ issume that the sum of geometric series $\sum_{n=0}^{\infty} aq^n$ equals value s, i.e. $\sum_{n=0}^{\infty} aq^n = s$. That means that the equation $s = a + aq + a^2q + ... + a^2q +$ is valid. Multiply the equation by a nonzero number *q*, so that we get $sq = aq + aq^2 + aq^3 + ... + aq^{n+1} + ...$ *n* $asq = aq + aq^2 + aq^3 + ... + aq^4$ $^{+}$ quation $s = a + aq + aq + ... + aq + s$ valid. Multiply the equation by a honzero number q, so
hat we get $sq = aq + aq^2 + aq^3 + ... + aq^{n+1} + ...$ By subtracting these equalities we obtain $s - sq = a$, wherefrom 1 $s = \frac{a}{a}$ *q* $=$ \overline{a} , which is the formula derived from the figures. of making light visible. To show the path of light in materials, you need special equipment – smoke t appears that the sum of infinite geometric series is dependant on the first term of the series as well issume that the sum of geometric series $\sum dq$ equals value s, i.e. $\sum aq = s$. That means that $n=0$ actually changes. We want to demonstrate how $n=0$ reflection and refraction – not *instead* of the actual experiment (if one sees experiments only in wherefrom $s = \frac{c}{\epsilon}$, which is the formula derived from the figures.

differences reaches the other surface of the lens – again mathematics is required to calculate the again mathematics is required to calculate the angle in \mathcal{L} \overline{a} and continues the glass and continues the same happens when the same happens whe

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- $[4]$ http://www.prof.jozef.doboš.eu/MA1.pdf (July 13, 2011)