Lifelong Learning Programme

DynaMAT

Finding geometric patterns as a game of dynamic explorations

Toni Chehlarova, Evgenia Sendova

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences

1 Introduction

Many interesting geometric problems deal with a locus - the set of points satisfying a particular condition. The traditional problems on loci are limited to finding simple curves. The dynamic geometry software allows for much more sophisticated explorations.

In this section we shall demonstrate how the *What-if* strategy could be applied in the context of:

- a traditional geometry problem
- a generalization of a well known problem
- an Olympiad problem in the practice of an experienced mathematics teacher (Appendix I)
- problems investigated by high-school students (Appendix II, Appendix III)

1.1 Looking at the classics with a dynamic eye

The problem:

> *What is the locus of the midpoints of the segments joining a fixed point within a circle with the points of that circle?*

To solve this problem you could study the behaviour of the midpoint under question while moving the endpoint of the segment on the circle along it.

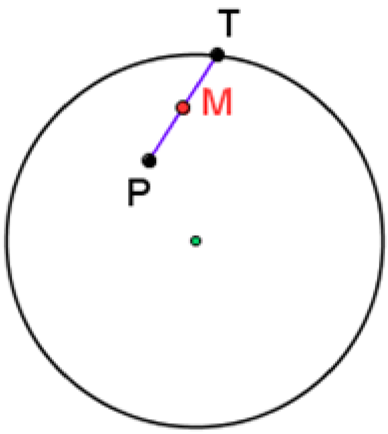
Let us start with a dynamic construction (say in *GeoGebra*):

Task 1. Construct a circle and a point **T** on it (by the **Circle** button or the **Circle** command)

Task 2. Then construct a point **P** within the circle (by the **Point** button or the **Point** command)

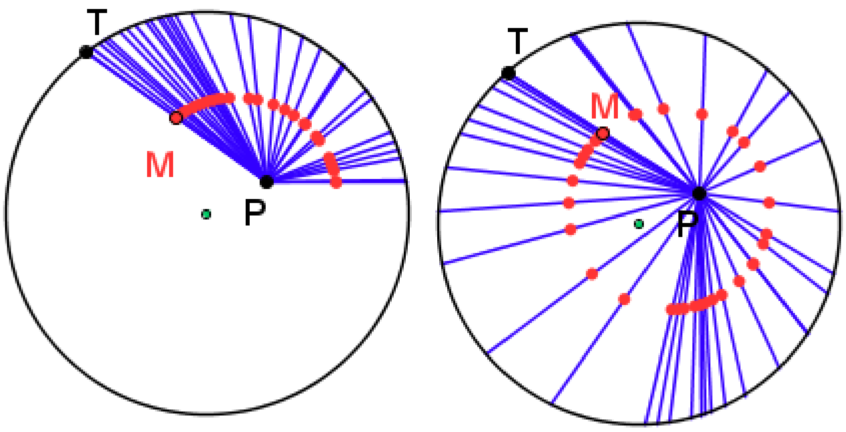
Task 3. Finally construct the midpoint **M** of the segment **PT** (by the **Midpoint** option or command)

The construction looks like the following one:



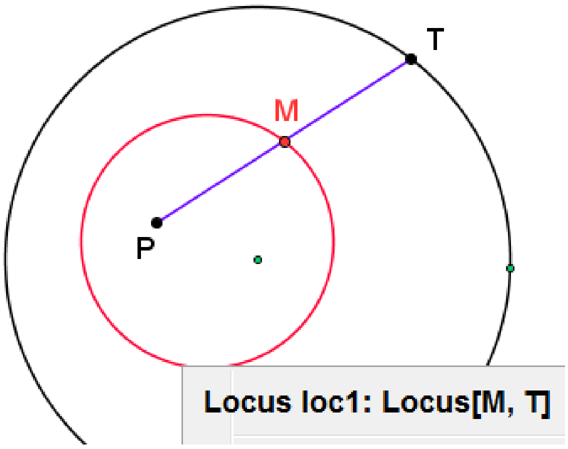
Task 4. Now observe the trace of **M**'s path. To do this put **M** (and why not the whole segment **PT**) in a **Trace mode** and drag **T** along the whole circle.

You will see that the shape formed by **M**'s sequential positions looks like a set of points on a circle.



There are various ways to strengthen your conjecture regarding the shape of the locus.

Task 5. Construct the locus of **M** by means of the **Locus** option (under the **Line** button) or the **Locus** command.

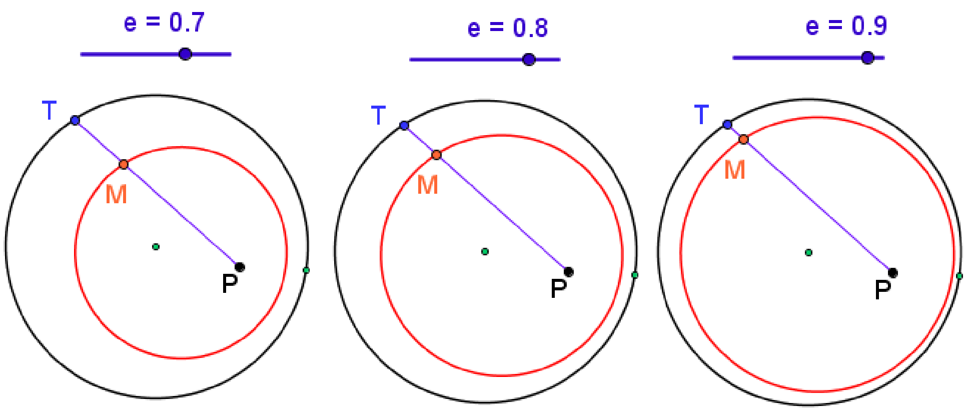


The game is not over, however. It is time to ask more *What-if* questions, e.g. *What if **M** is not the midpoint, but divides the segment at a fixed ratio? What if **P** is outside of the circle?.*

You might think that in this case the locus would look like a more general curve of a second degree – an ellipse perhaps?...

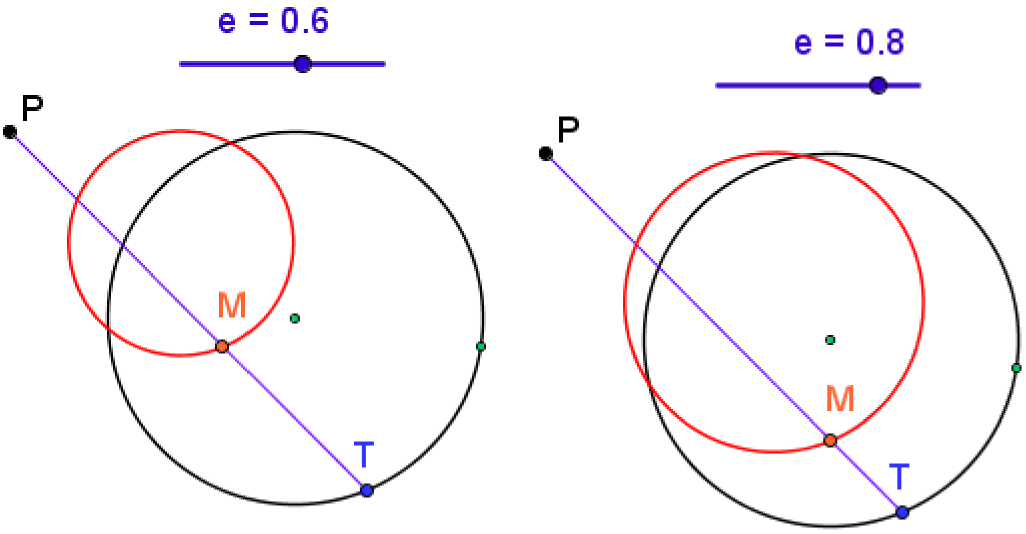
There is a good way to generalise the explorations:

Task 6. Add a slider for the ratio **e** in which **M** is dividing the segment, i.e. **PM** = **e PT** and explore for various values of **e**:



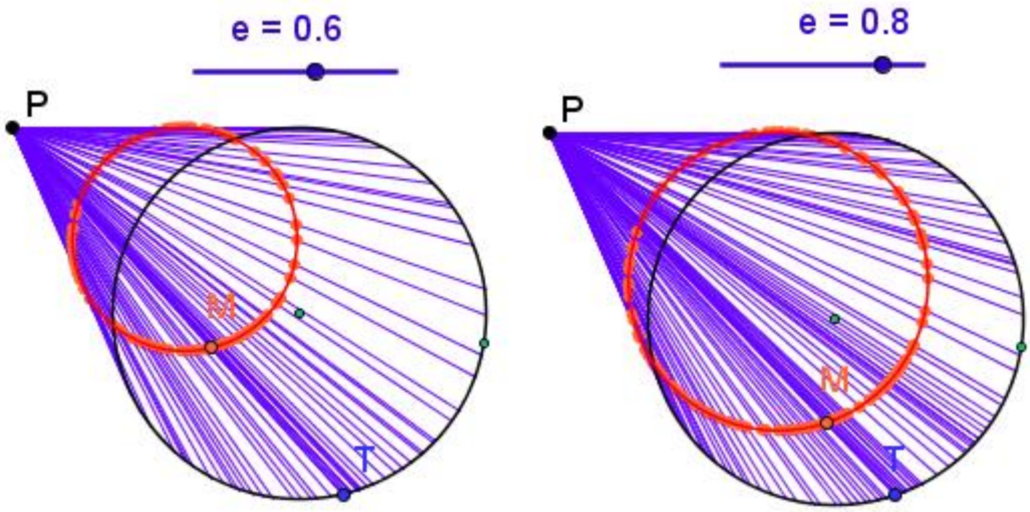
The locus seems to be circle again!

Task 7. Check what happens if the point **P** is outside of the circle.



A circular shape once again!
The **Locus** command gives a pretty good idea about the shape of the locus but it works for a set of points only. What if you decide to visualize the set of segments for a point outside of the circle?

Task 8. Turn on the **Trace mode** for the segment **PT** when **P** is outside the circle and move **T** for various (and then fixed) values of **e**:



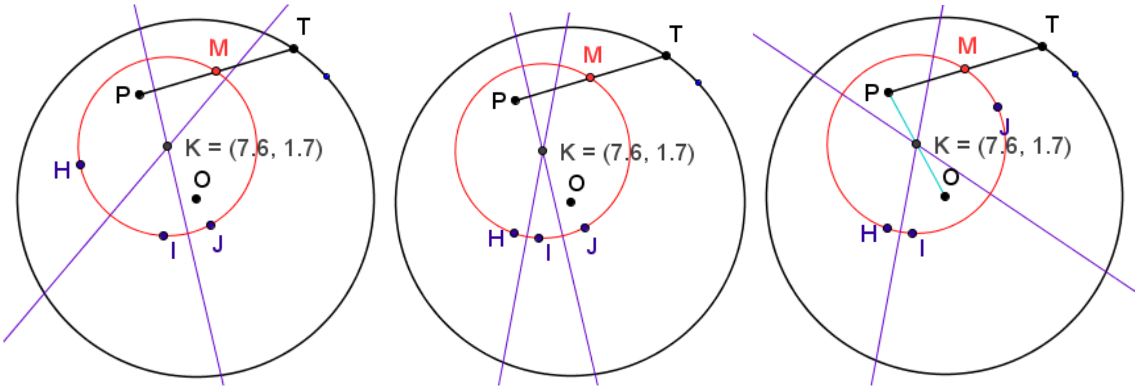
An interesting visual effect! But could you be absolutely sure that **M** describes a circle? What if it is in fact an ellipse which is very close to a circle…

Before reading further try to think of possible ways to verify your conjecture.

One way to do this (still experimentaly) is to construct 3 points (**I**, **J** and **H**) on the locus, pass a circle through them and check if this circle concides with the locus.

Another way which could help you prove the conjecture rigorously is to observe some interesting properties of the construction enriched with some auxiliary elements:

Task 9. Construct 3 points (**I**, **J** and **H**) on the locus, then the perpendicular bisectors of the segments **IH** and **IJ,** and finally - their intersection point **K**. Now move some of the points **I**, **J** and **H** and observe the coordinates of **K** as well as the length of **KJ**, **KI**, **KH.** What is the relationship between **K** and the center **O** of the original circle?



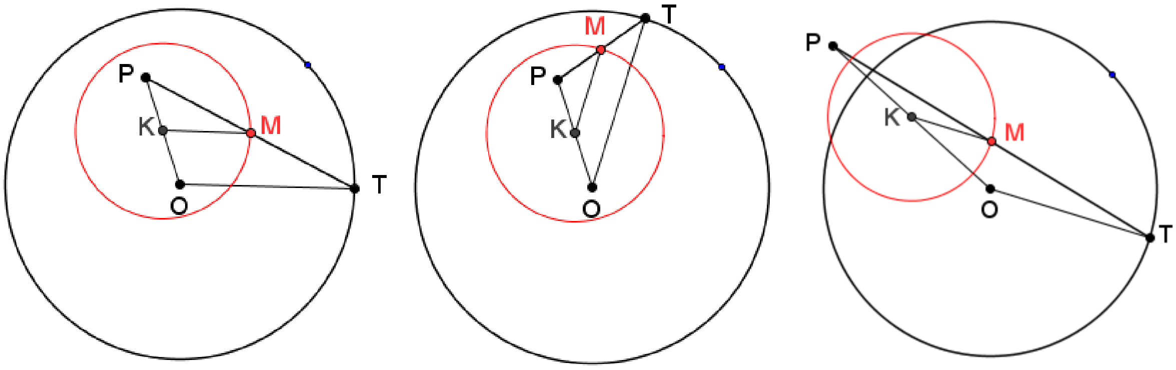
Yes, it is easily seen that **K** keeps its coordinates the same. In addition **KJ** = **KI** = **KH** which shows experimentally (but **with a greater degree of conviction**) that the locus is a circle.

Furthermore, you should have noticed that the center **K** of the locus is the midpoint of **PO**, where **O** is the center of the original circle.

Now we are ready to prove rigourosly that the locus is a circle with a center the midpoint **K** of **PO**, (where **O** is the center of the given circle), and a radius - half of the radius of the given circle.

The proof

Let **K** be the midpoint of **PO**. Then $\mathbf{KM} = \frac{1}{2}\mathbf{OT}$, i.e. the midpoint **M** is at a constant distance to **K** (a half of the radius **OT** of the given circle) while **T** is moving along **O**.



Therefore the locus is a circle with a center **K** and a radius – half of the radius of the given circle.

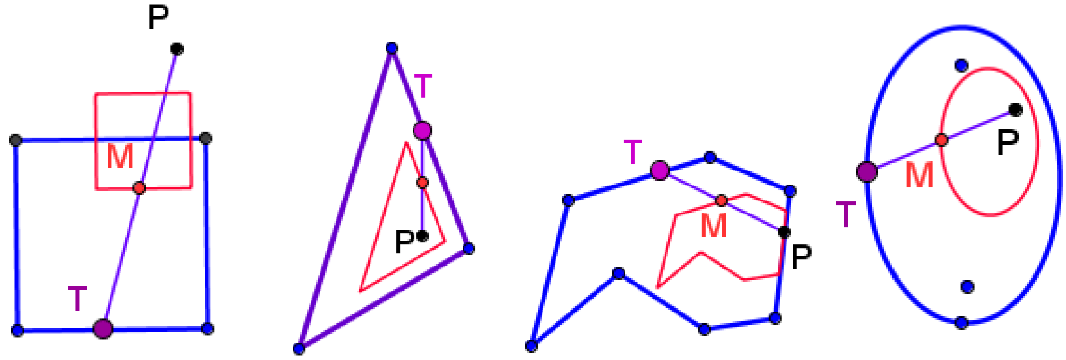
Task 10. Prove the theorem in the general case for **PM** = **ePT** .

Task 11. Replace the circle with:

- a square
- a triangle
- an arbitrary regular polygon
- an ellipse
- a curve of your own choice

Task 12. Generalize your finding

If your students have studied *dilation* (in the Bulgarian curriculum it is introduced a year after the first occurrence of *loci*) they could use it to solve the problems but it is very appropriate to generalize their findings (Task 12) after getting the results from Task 11:



Applying the *What-if* strategy could cultivate an exploratory spirit in mathematics classes - the students are encouraged to explore interesting partial cases, to generalize relatively simple problems in various directions, and even to attack and generalize challenging problems of Olympic level [Appendix I]

2 Generalizing a well known problem

In this section we demonstrate a process which is typical for the working mathematicians – we generalise a well-known problem, then we attack it with tools we believe are the most appropriate for the purpose (in our case with dynamic constructions we have specially designed in a *step-by-step refinement and enrichment* spirit). We try to systemize our explorations and we reflect on the ideas we get. It is the very process that will be of our primary interest rather than the description of the results. In addition, we shall be happy if you, the readers, get motivated to attack some of the open problems yourselves.

Here is our starting point:

A well-known problem

Find the locus of the centers of the equilateral triangles inscribed in an equilateral triangle.

We expect that this problem is known to most of our readers. An ambitions generalization would be:

An ambitious generalization

Find the locus of the centers of the regular ***m****-gons inscribed in a regular* ***n****-gon,* $\boldsymbol{m \le n}$*.*

Further below we shall write **(*m*;*n*)** to denote the construction of a regular ***m***-gon inscribed in a regular ***n***-gon. Note that we are not even sure for which ***m*** and ***n*** the **(*m*;*n*)** constructions are possible.

Let us start our *attack* with a more modest problem, dealing with the case **(3;*n*)** for ***n*** = 3, 4, …

The first attack – the (3;*n*) case

Find the locus of the centers of the equilateral triangles inscribed in a regular ***n****-gon.*

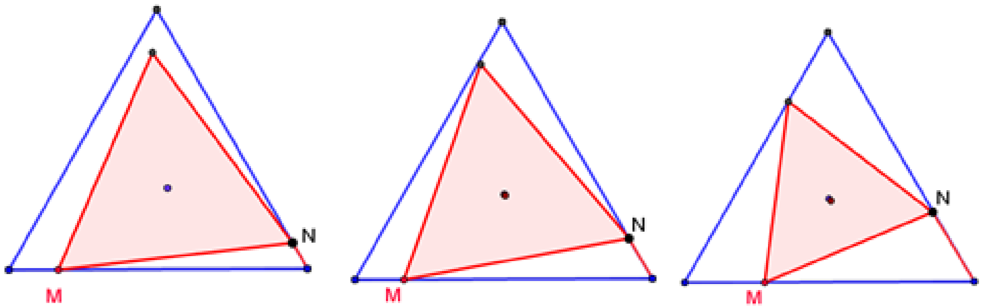
2.1 A primitive (hand-made) dynamic model

We construct an equilateral triangle two of whose vertices are on the ***n***-gon and move the third one so as to get an inscribed triangle**.**

But first things – first! To get the flavor of the dynamic construction which could be then generalized it is natural to start with the simplest case (***n*=3**), and proceed in what could be called a *hand-made* mode:

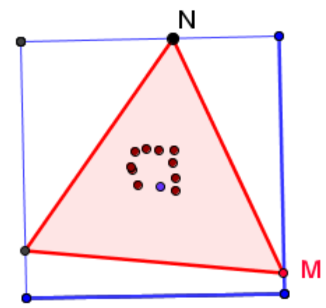
- We select two arbitrary points **M** and **N** on different sides of the given (the *blue*) triangle
- Then we construct an equilateral (*red*) triangle with a side **MN** (it doesn't matter which one of the two).

- Next we move point **N** (keeping **M** at irs current position) so that the *red* triangle becomes inscribed in the blue one. The center of the red triangle is a point of the locus we seek
- Now repeat the above proces for a new position of **M.**

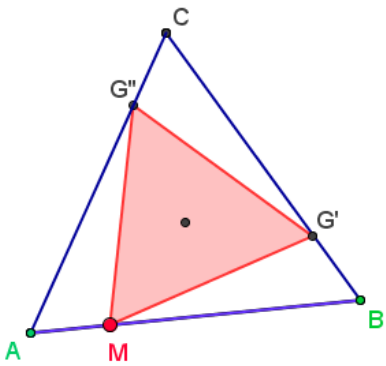


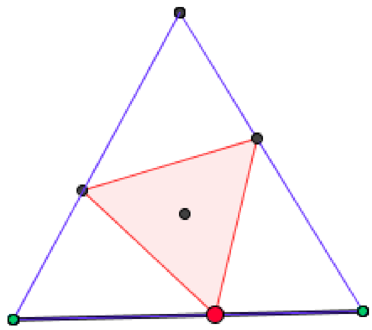
Thus, using consecutive positions of point **M** we get an approximate idea about the locus - in the **(3;3)** case the centers seem to coincide (or are perhaps close enough)…

If we apply a similar procedure for the **(3;4)** case the centers appear to be on a square. But inscribing the triangle *by hand* is a time-consuming method. (Still, slightly better than constructing by hand on a paper and considering just one case which might be misleading due to the imprecision [Appendix II]).



To automize the construction let us take a better look at the **(3;3)** construction. It is natural to conjecture that in this case the locus is a single point coinciding with the center of the given triangle.





The congruence of the triangles **AMG"** and **BG'M** implies **AM**=**BG'**. Therefore, we can use in this particular case a dynamic construction based on the congruence.

2.2 An automized dynamic model for (3;3) constructions

Here are various methods of creating automized models for the **(3;3)** constructions:

The first method

- we construct a point **M** on the contour of a regular **3**-gon
- we construct a circle ***k*** with center **B** and radius **AM**

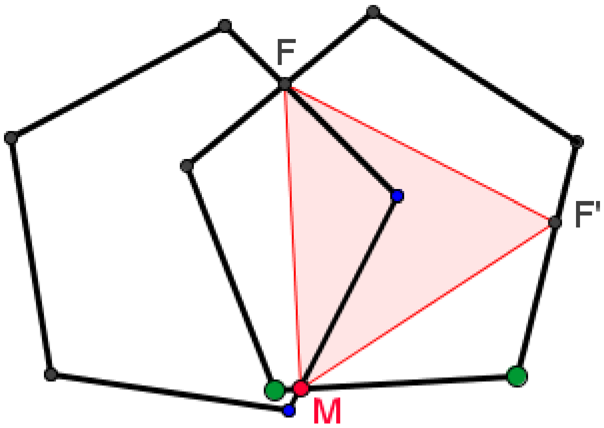
- we denote by **G'** the intersection point of ***k*** and the side **BC** of the ***n***-gon (in this case the triangle **ABC**)
- we construct **G''** in a similar way
- we connect the points **M, G'** and **G''** in a triangle.

The second method

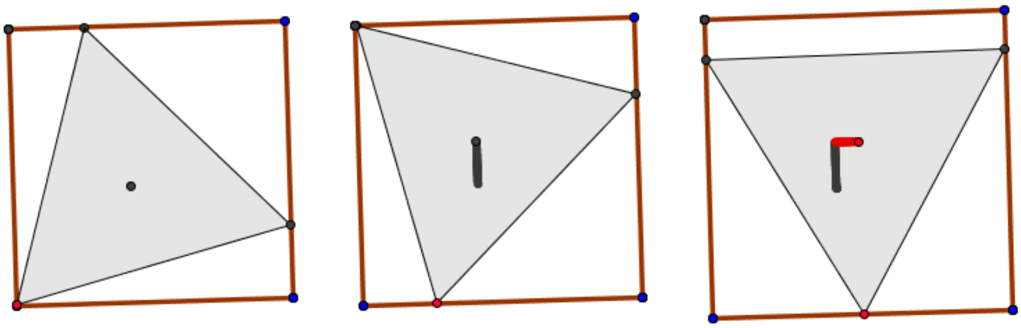
- we construct a point **M** on the contour of a regular ***3***-gon`
- we construct the image **G'** of **M** under rotation with center – the center of the given triangle and angle 120^o
- Then we construct the image **G''** of **G'** under rotation with center – the center of the given triangle and angle 120^o
- we connect the points **M, G'** and **G''** in a triangle

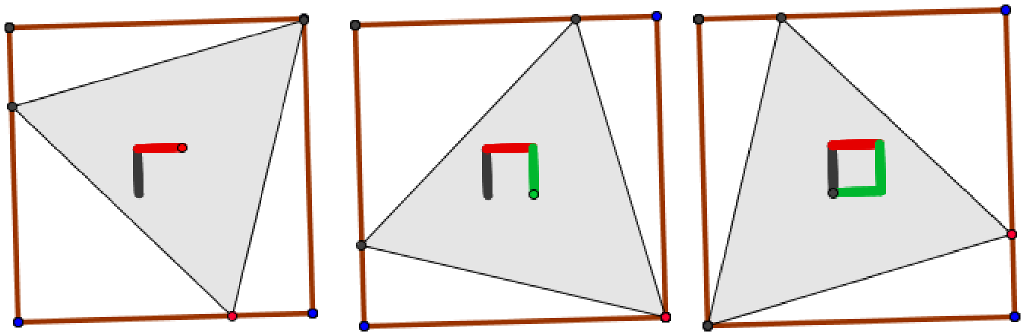
2.3 More Dynamic Models

- We construct a point **M** on the contour of a regular ***n***-gon.
- Then we construct the image of the ***n***-gon under rotation **ρ** with center **M** and angle 60^o.
- We construct their interestion point be **F.** (It will be another vertex of the equilateral triangle whose first vertex is **M**, and which is inscribed in the ***n***-gon.
- Then we construct the thrid vertex as the pre-image **F'** of **F**.
- We connect **M**, **F'** and **F** to get the equilateral triangle inscribed in the ***n***-gon.



Here are some snap-shots of the trace the triangle's center in the [(3;4) construction] leaves during the movement of the inscribed triangle:

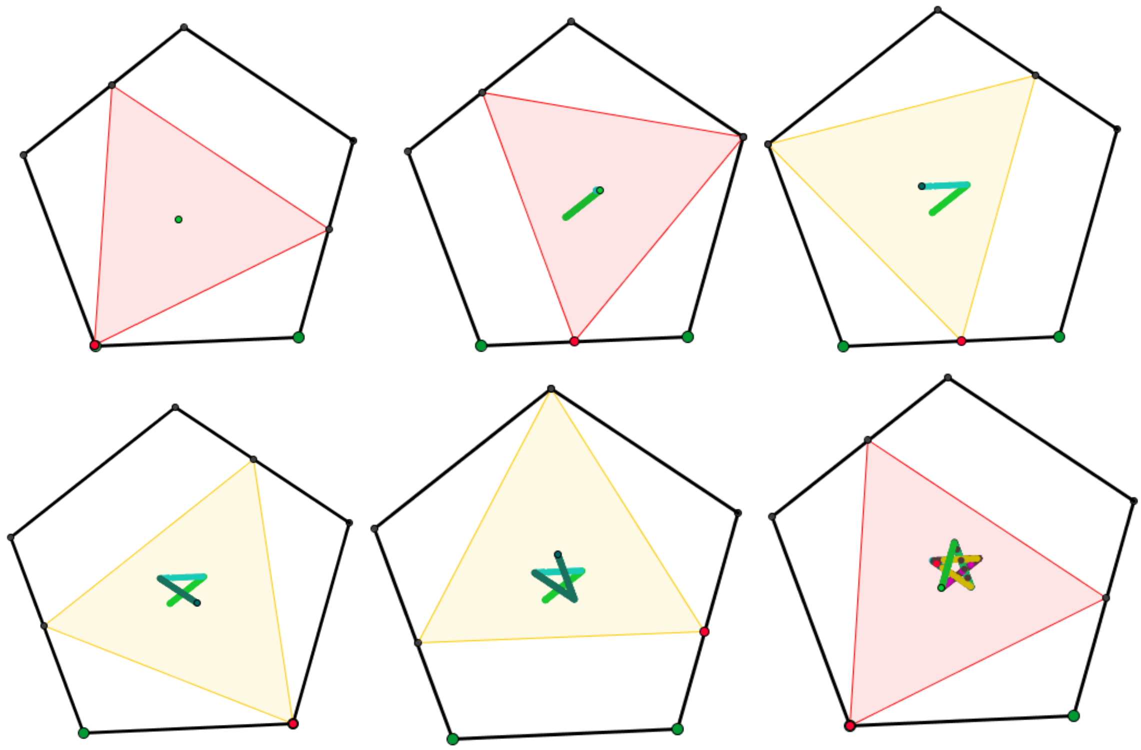




When we move the red point (**M**) until the next vertex of the triangle coincides with a vertex of the square (i.e. takes its initial position) we observe the trace becoming a shape which looks as a half of square. By analogy, when moving the point **M** along the rest of the sides of the square the center of the triangle will leave a trace which completes a square-like shape and after which it will start repeating the trace (three times).

If the considered locus of the **(3;4)** construction is a square indeed could we conjecture that the corresponding locus of the **(3;5)** construction would be a regular pentagon?

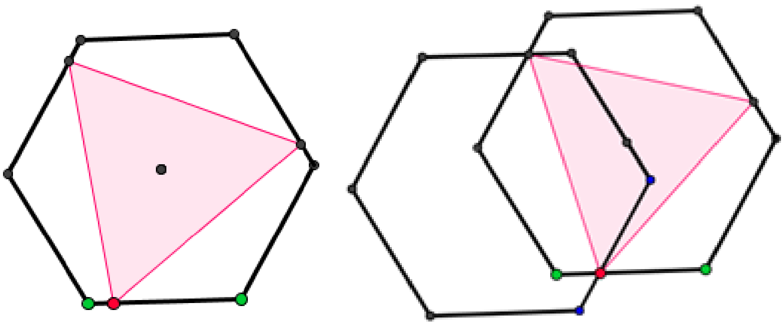
In the latter case it is sufficient (again due to the symmetry) to observe the effect of the movement of the red point on a part of the pentagon only.



A-a-ah! Still 5 sides but it does not look like a pentagon – rather like a pentagram! Then what we suspected to be a square could be considered maybe as a „4-side star“...

Again, the center of the triangle describes the locus three times while the red point makes a full round along the original pentagon.

In the **(3;6)** case the locus appears to be a single point:



Such was the locus in the **(3;3)** case. By analogy we could conjecture that the same would hold for **(3;9),** and more general – for **(3;3*k)***

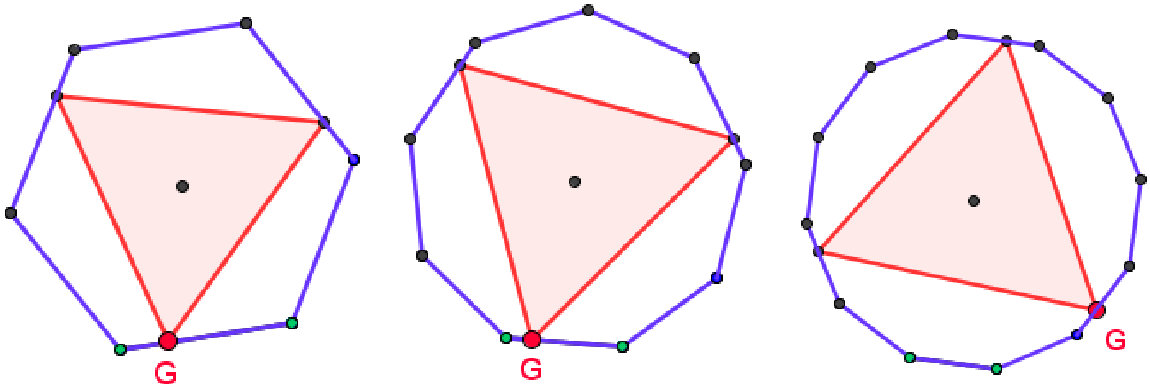
We could make separate construction for the **(*m*;*km*)**.

2.4 Further explorations providing insight

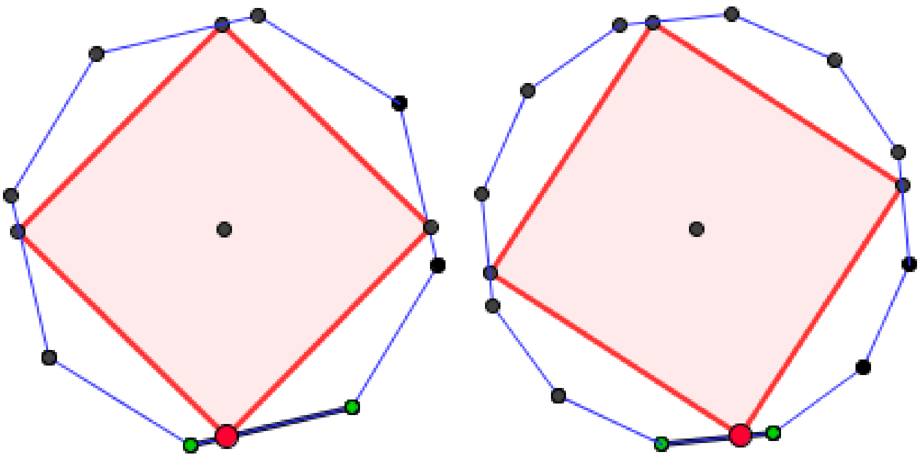
The (*m*; *km*) model

Now we are tempted to explore further

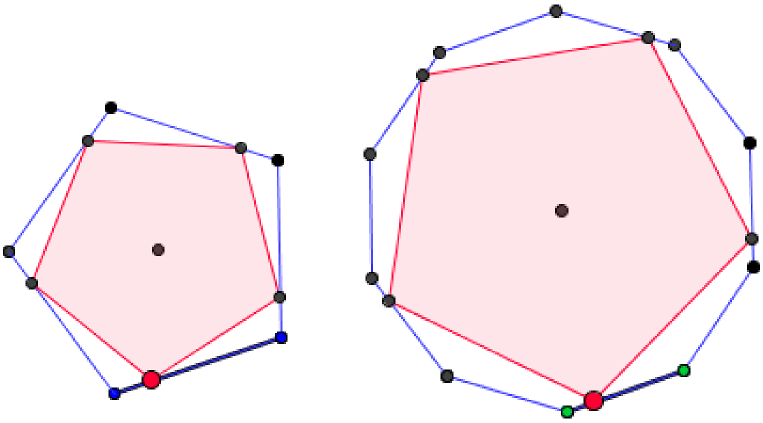
(3;3k**)**



(4;4*k*)



[(5;5*k*)](#)



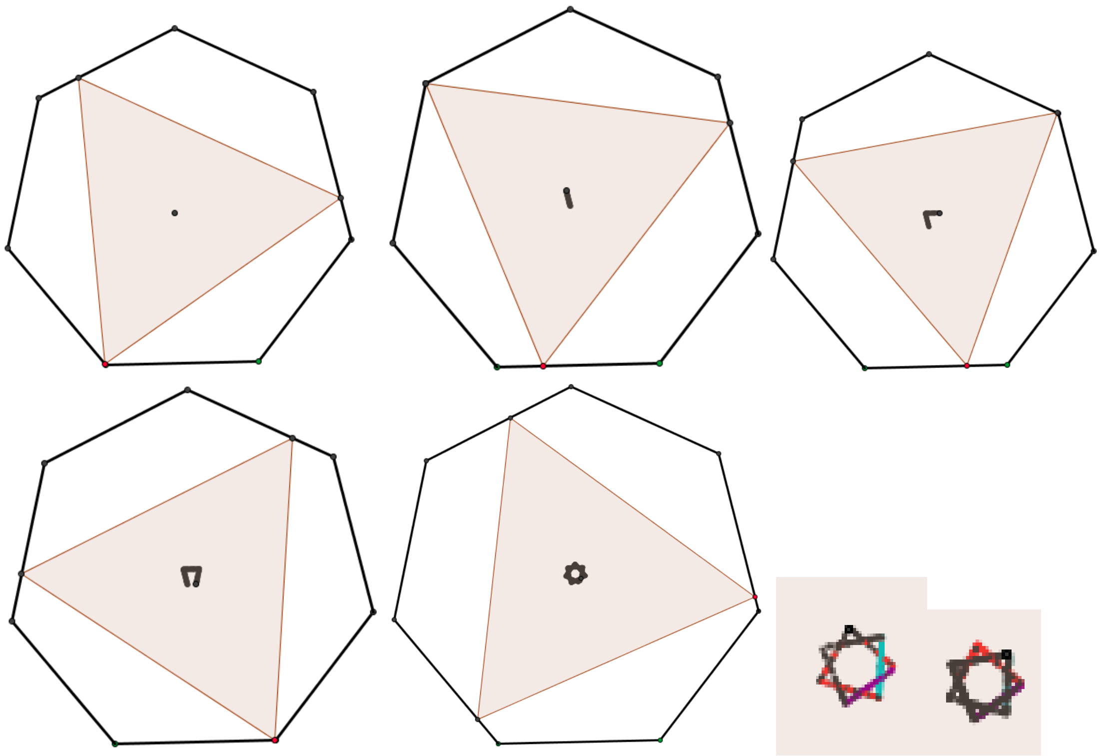
The general conjecture we could draw after exploring the **(*m*;*km*)** model is that *for every point **G** on the **n**-gon (**n**=**km**) there exists an inscribed **m**-gon with a vertex **G** and the locus under consideration is a single point coinciding with the center of the **n**-gon.*

The **(*m*;*km*)** constructions could be also achieved by analogy of the methods in 2.2.

Let us continue our explorations with the (**3;*n***) model.

The (3;7) model

Now we are expecting a star with its generating module emerging when going along one of the heptagon's sides.

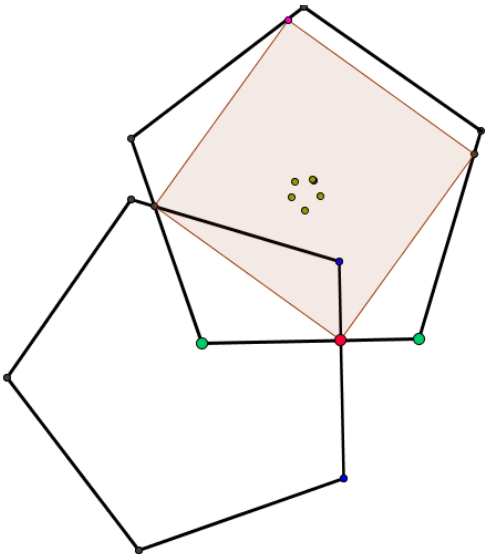


Exploring the model (**3;*n***) leads us to the conjecture that it is possible to inscribe an equilateral triangle in every regular ***n***-gon. In other words, that (**3;*n***) *is always a possible construction*.

It is interesting to see what is the situation in the case of the (**4;*n***) model...

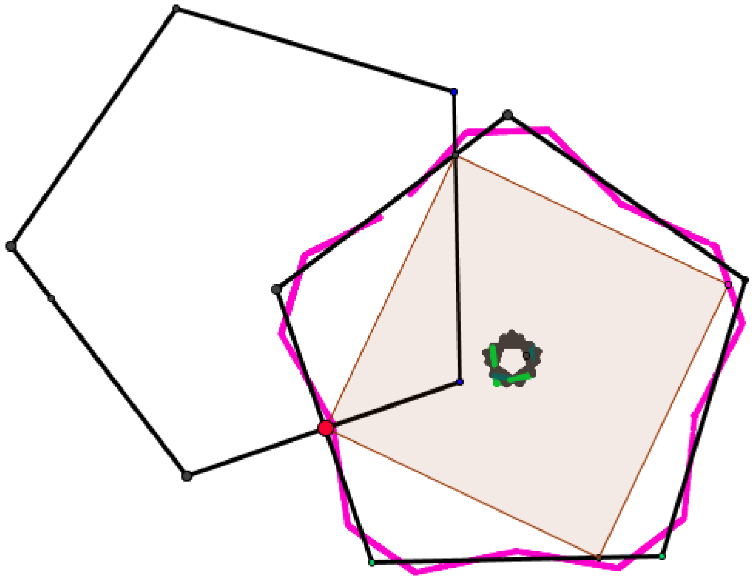
The (4;5) model

This is in fact an inscribed square in a regular pentagon:

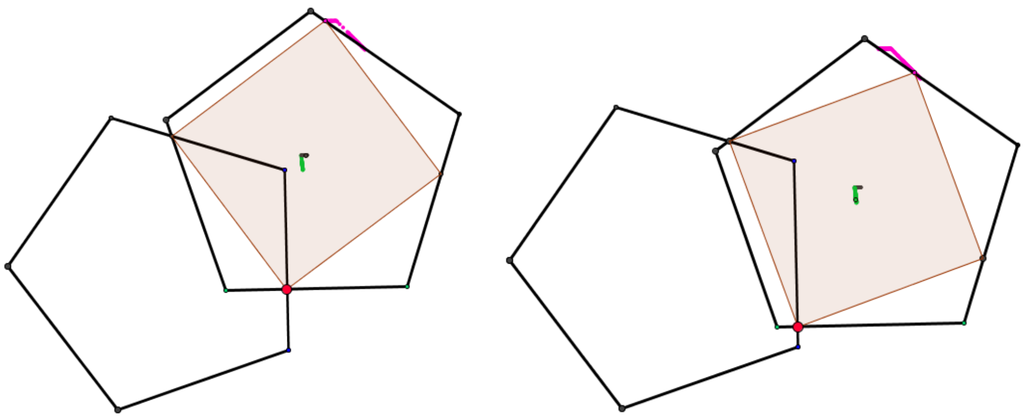


Here again the given *n*-gon (pentagon in this particular case) and its image under 90^{o} rotation have a single intersection point whose pre-image is the third vertex of the square. What is left is to check when the fourth vertex of the square is on the given ***n***-gon,

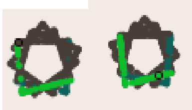
Here is the fourth vertex in purple (in a *trace model*):



Below are several positions leading to an inscribed square:

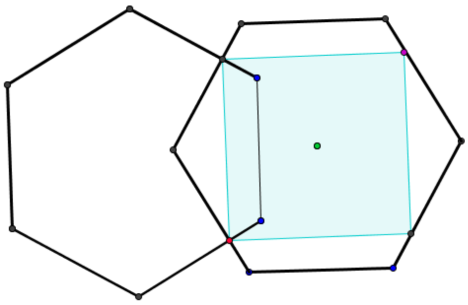


Let us take a better look at the trace of the center of this square with three vertices on the ***n***-gon



In fact, this is the locus of an inscribed rectangular isosceles triangle in a regular pentagon.

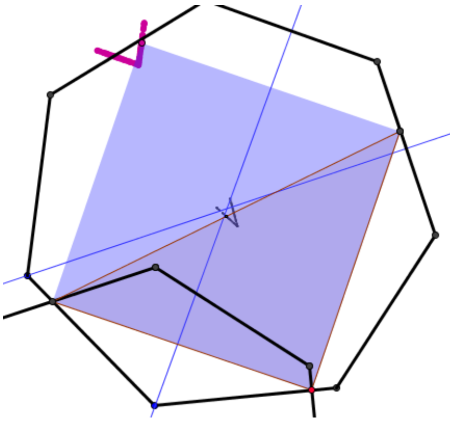
The (4;6) model

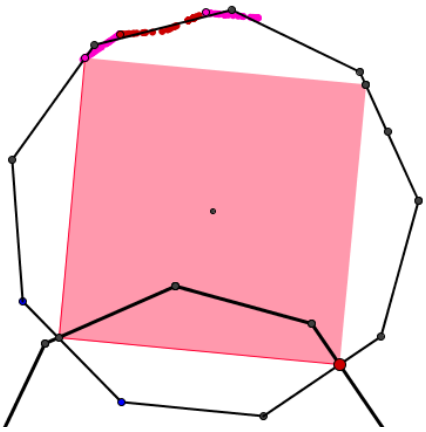


The locus is a single point again as in the case of (***m, mk***) the difference being that this time *it has been formed as a locus of a finite number of inscribed squares.*

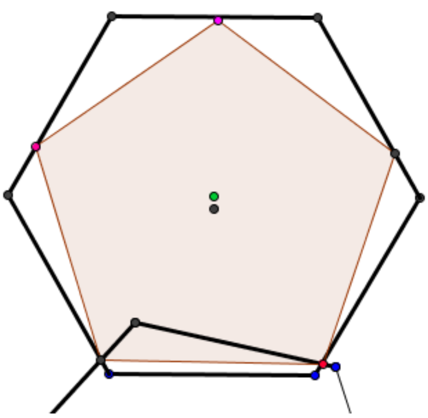
At this point it would be a good idea to guess how many points would the locus of the **(4;7)** and **(4;9)** constructions contain. Let us check experimentally:

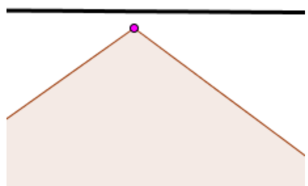
The (4;7) and the (4;9) models

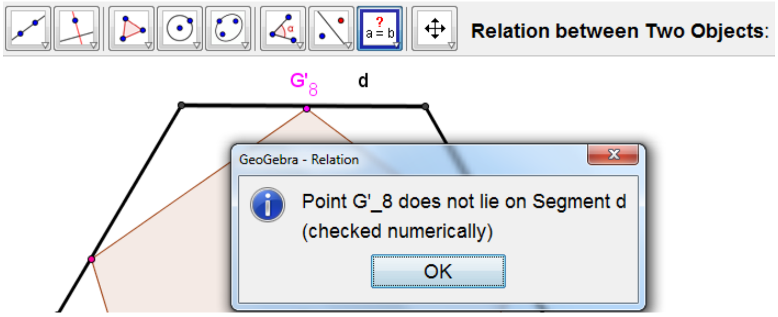




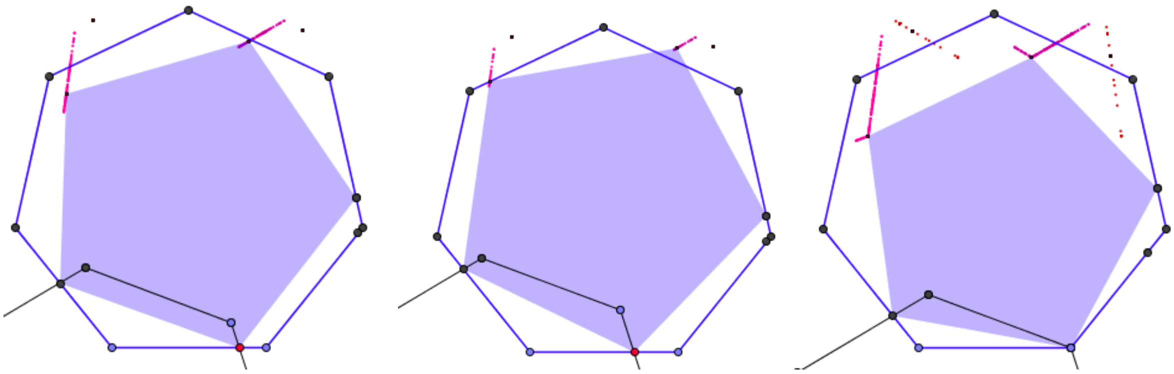
In the **(5;6)** model for explorations it appears at first glance that the fifth vertex is on the hexagon. We could check this experimentally by zooming the screen or by checking two objects for coincidence. But even after a positive answer we should not forget that the computer works with a certain (finite) precision.

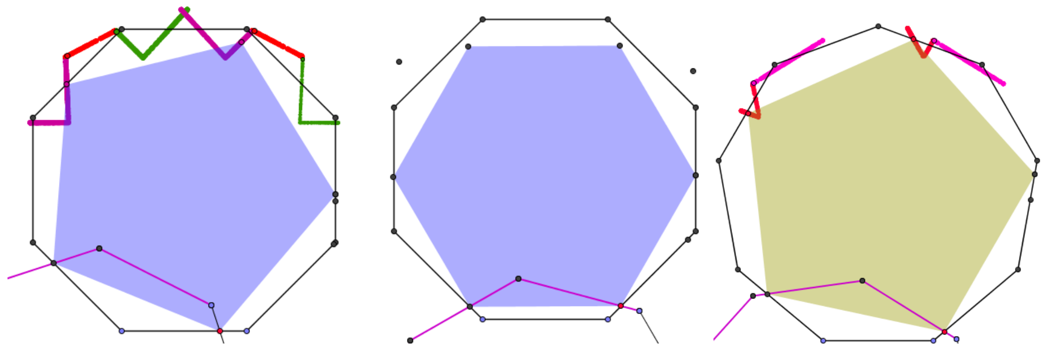




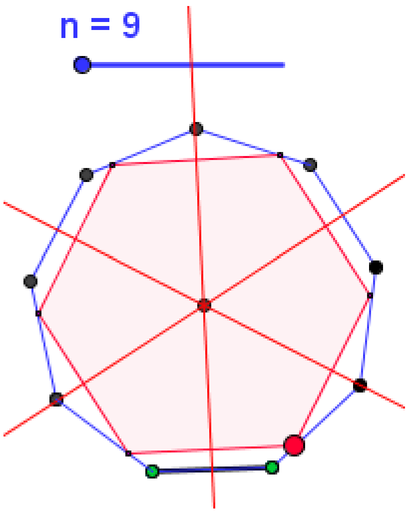


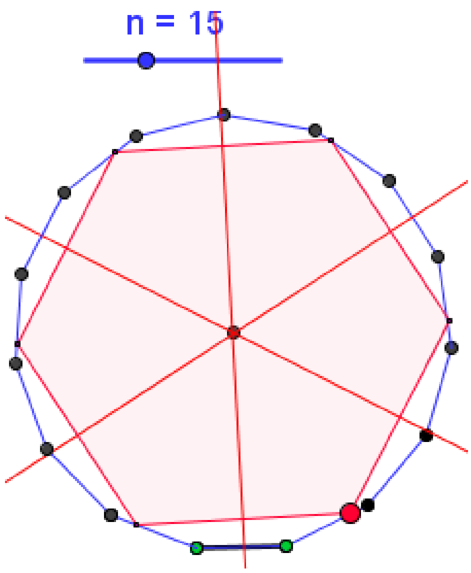
Here are some dynamic models in search of constructions, which experimentally lead us to conjecture that they are impossible or at least doubtful.



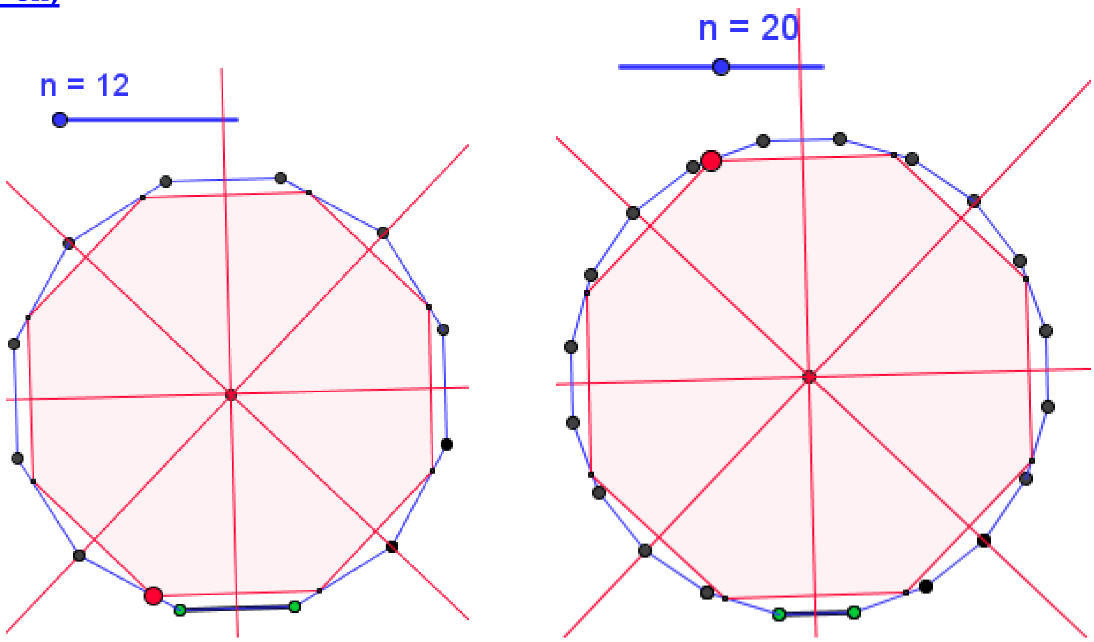


However, some special cases like the ones below could be constructed in view of the symmetry: **(6;3+6k)**





[(8;4+8k)]



For even **m** the line symmetry yields equality of the segments in construrctions of the kind $\left(\mathbf{m};\frac{\mathbf{m}}{\mathbf{2}}+\mathbf{km}\right)$.

At this point it is a good idea to stop and take a look around – *what is known in relation to our explorations?* We entered the magic phrase ***a regular m-gon inscribed in a regular n-gon*** and here it appeared [1]! Almost the same title and the same denotation showing how natural it is in its simplicity and conciseness when exploring various cases and describing the conjectures and results. The authors Dilworth and Mane present there the necessary and sufficient conditions on ***m*** and ***n*** for inscribing a regular ***m***-gon in a regular ***n***-gon. It is interesting to note that *naively* (their own phrasing) they expected *this problem to be solved in the time of Euclid, but it seems to be not completely solved.*

Here is what Dilworth and Mane prove in [1] by means of complex numbers:

> **Theorem.** *Suppose that* $m, n \geq 3$. *A regular m-gon can be inscribed in a regular n-gon if and only if one of the following mutually exclusive conditions is satisfied:*
> (a) $m = 3$;
> (b) $m = 4$;
> (c) $m \geq 5$ *and m divides n;*
> (d) $m \geq 6$ *is even and n is an odd multiple of* $m/2$. *(Note that this includes the case* $n = m/2$.*)*

It turns out that the last examples of our explorations belong to (d). (Note that [1] includes the case **m** > **n**.)

Had we seen this article before attacking it with dynamic means we would feel very reluctant to offer it to students (even if they were very motivated to explore new mathematical territories). However, the explorations themselves harnessed mathematical skills accessible to students knowing about geometric transformations. Furthermore, the patterns and the relationships observed during these explorations gave rise to other interesting questions.

What really matters for us in relation to this problem is not even the solution itself but the whole process of creating a good platform for explorations, enhancing our intuition and understanding about some patterns among the constructions, designing a more systematic approach of explorations, realizing that not all combinations of inscribing a regular ***m***-gon in a regular ***n***-gon are possible, and finally – the belief in teachers' ability to promote the inquiry-based learning of mathematics. In a nut shell, to illustrate the „grook" [2] of the great Danish mathematician, architect and poet Piet Hein:

Problems worthy of attack, prove their worth by hitting back.

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Reference

[1] Dilworth S. J., S. R. Mane. Inscribing a regular *m*-gon in a regular *n*-gon http://www.math.sc.edu/~dilworth/preprints_files/DilworthManeJOGpublished.pdf (October 25, 2011)

[2] *Grooks*, http://www.archimedes-lab.org/grooks.html (October 25, 2011)