Appendix I

An Olympic problem as a source of inquiry-based learning

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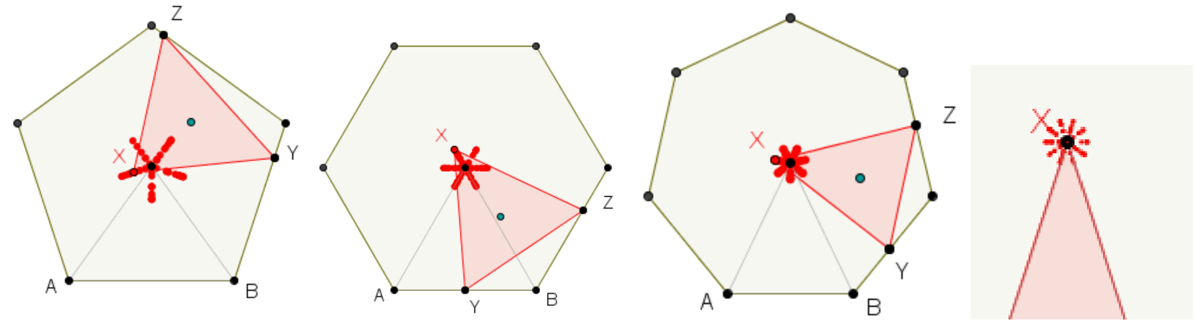
The problem under consideration was given at the XXVII Mathematics Olympiad, 1986, in Warsaw, Poland. Two years later it was proposed for further investigations with the language-based software *Geomland* [1] to me and my colleague Ms. Tanya Slavcheva by our mentor Dr. Ivan Tonov as part of our diploma thesis [2]. Here is its formulation.

The problem

Let A and B be adjacent vertices of a regular N-sided polygon (P), N≥5, centered at the point O. The triangle XYZ, which is congruent to the triangle OAB and initially coincides with it, moves in the plane of the polygon in such a way that the points Y and Z remain on the boundary, while the point X is in the interior of P. Find the locus of X.

From the results we observe on the computer screen we cannot draw an obvious conclusion. We can see a figure resembling a "snowflake". In other words, the set of points traced out by the point X consists of N segments with common end point O, the center of the polygon. Each segment points from point O towards the polygon's edge whose number is $\frac{N+4}{2}$ for an even N, and perpendicular to the segment between point O and the polygon's edge whose number is $\frac{N+3}{2}$ for an odd N. Moreover, the point X moves away from point O a certain distance along the respective segment, after which it starts moving towards the point O.



Many of the students encounter difficulty understanding the motion, when described as above. This is where the dynamic geometry software provides great help in visualizing the process.

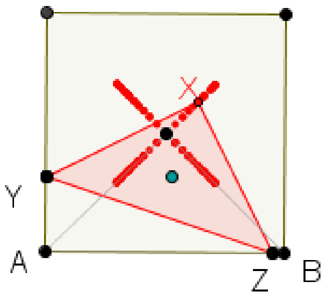
These observations can lead to the discovery of other relationships between the geometric objects:

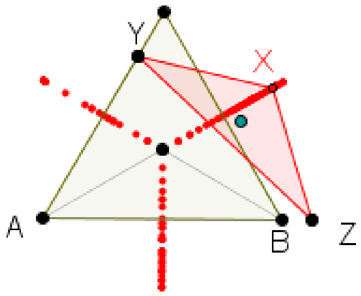
1. One can note that the number of segments forming the "snowflake" equals the number of edges of the polygon, and that adjacent segments meet at an angle equal to the central angle of the polygon.
2. An increase of the number of edges of the polygon leads to a decrease of the length of the segments forming the "snowflake".
3. The question concerning the length of a "snowflake" segment arises naturally, i.e., we would like to find when the point stops going away from the center and turns back.

4. One might also be interested in the angle between the segment XY and the side of the polygon containing Y, as well as the angle between the segment XZ and the side of the polygon containing Z at the moment when X is the farthest away from O. It turns out that both of these angles are right.

As suggested by the experiments, the vertices of the "snowflake" can be constructed by intersecting a line passing through a vertex and the center of the polygon, and a line parallel to the edge containing the point Y, which is a distance R (towards the center) from the point Y.

Although the problem was posed for N≥5 the construction discussed here allows us to experiment with the cases N=4 and N=3 as well. While the case N=4 leads to the same conclusions, in the case N=3 one can observe that when Y moves along one side of the triangle (e.g., AB), Z moves along the continuation of the adjacent side BC, reaching a certain point and then coming back at C. Moreover, the maximum length of the "snowflake" segment is again achieved when the segment XY is perpendicular to AB. However, unlike for N≥5, the segments are not traced out twice.





These are in fact new problems, even though the implementation of the solution strategy is the same. Moreover, the experiments lead us to new problems. For example, in the case N=3 we can pose the following problem**:**

A modified problem

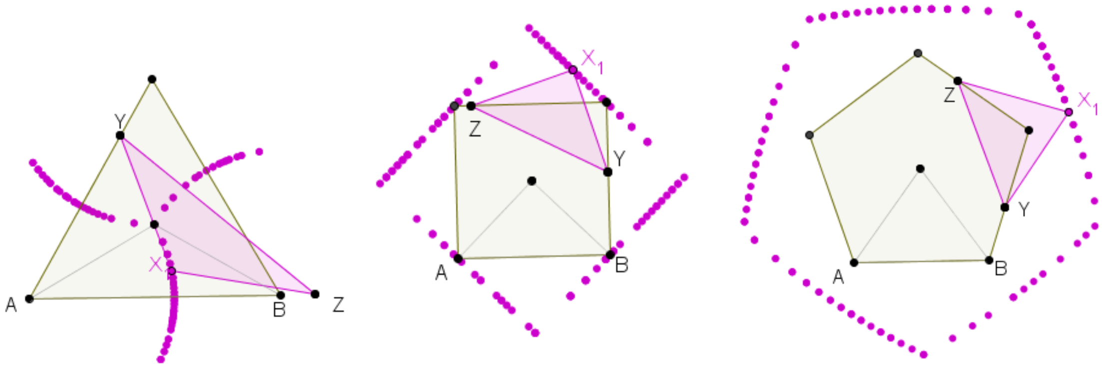
Let A, B and C be the vertices of an equilateral triangle, and O be its center. A triangle XYZ, congruent with OAB, starts out in the following position: X is outside the triangle ABC, Y=A, Z=C. The triangle XYZ moves as follows – the point Y traces out the sides of the triangle ABC, while the point Z traces out the extensions of the triangle sides, adjacent to the sides containing Y. Find the locus of X.

The solution to this problem is analogous to the solution of the original problem. The length of each of the "snowflake's" three segments is equal to the radius R of ABC's circumcircle.

Naturally, one is curious about the following experiments.

Experiment 1. In the case N=3, the triangle XYZ moves from its initial position X=O, Y=A, Z=C as follows. The point Y traces out the sides AB, BC, and CA of the triangle ABC, while the point Z moves along the continuations of the sides BC, CA, and AB, respectively.

After performing the above experiment, one can see that the set of points traced out by the point X, consists of three arcs of the circles centered at the vertices of the triangle ABC, of radius R (where R is the radius of ABC's circumcircle). Each of the arcs has arc length $\frac{\pi}{3}R$.

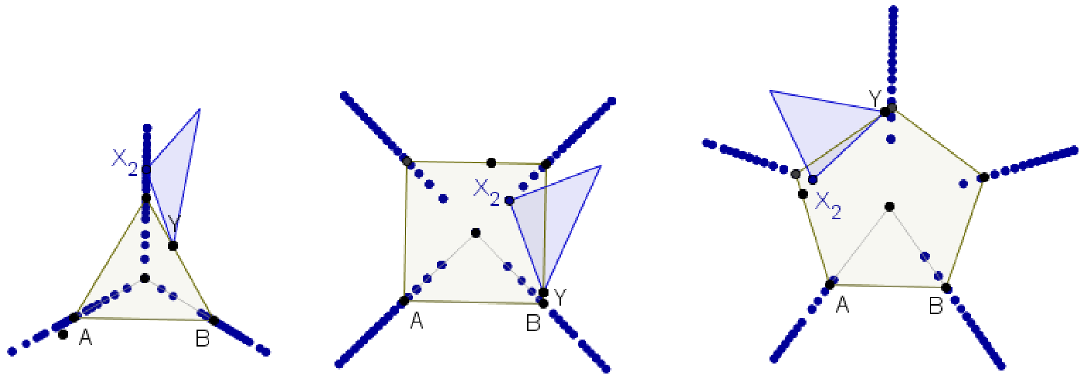


In the case N=4, the triangle XYZ moves from its initial position X=I_4, Y=A, Z=B, where I_4 is the point symmetric to the center of the square ABCD with respect to AB, as follows. The point Y traces out the sides AB, BC, CD, and DA of ABCD, while the point Z moves along BC, CD, DA, and AB, respectively. The set of points traced out by the point X consists of segments, passing through the square's vertices and parallel to and equal in length with its diagonals.

In the case $N \geq 5$, the initial position of the triangle XYZ is as for the case N=4. The set of points traced out by the point X consists of parts of higher degree curves. In general, with the exception of the case N=3, the set of points is traced out in a continuous way.

Experiment 2. In the case N=3, the triangle XYZ starts its motion from initial position X=O, Y=A, Z=B. The point Y traces out the sides AB, BC, and CA of the triangle ABC, while the point Z moves along the extensions of the sides BC, CA and AB, respectively. Unlike in Experiment 1, Z moves along the ray staring at B, C, and A, respectively, away from the triangle sides.

The set of points traced out by the point X, consists of three segments, each of which has length 2R. Here R is the radius of ABC's circumcircle. The center O of the triangle is one of the end points of each of the segments, the other end points are the symmetric points of the vertices of ABC with respect to O.



Analogously, in the case of a square, in this experiment, the set of points traced out by X consists of four segments of length 2R, where R is the radius of the square ABCD's circumcircle. The center O of the square is one of the end points of each of the segments, the other end points are the symmetric points of the vertices of ABCD with respect to O.

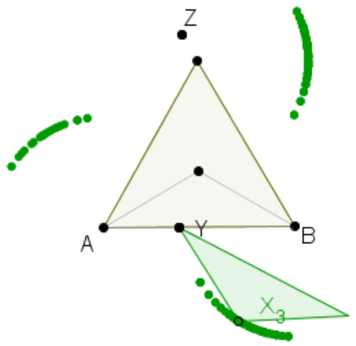
In the case N=5, 6, 7, etc., the motion of the triangle XYZ is as in the previous two cases. The set of points traced out by X consists of the center O of the polygon and N segments of length $2R - 2R\cos\frac{2\pi}{N}$. Each of the segments lies along a line through the point O and a vertex of the polygon. For each of the segments, one of the end points is a distance $2R\cos\frac{2\pi}{N}$ from the center O, and the other is the point, symmetric to O with respect to the corresponding vertex.

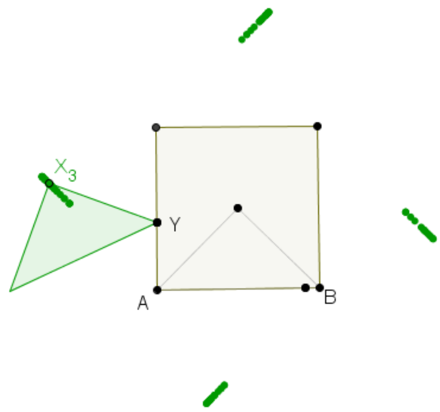
What stays the same in this experiment for various values of N?

- The segments, which constitute the set of points traced out by X, are traced out only once, unlike in the case of the original problem.
- The point Z moves in such a way that it traces out segments along the extensions of the respective sides of the polygon. Moreover, these segments are traced out twice.
- The point X is the farthest away from the center when the segment XY is perpendicular to the side containing Y.
- The set of points is always traced out in a discrete way. In the cases N=3 and N=4 the set is connected, while in the cases N=5, 6, 7, etc., the set of interest consists of the center O of the polygon and segments with empty intersection.

Experiment 3. In the case N=3, the triangle XYZ starts its motion from initial position Y=A, Z=B, and X coincides with the point symmetric to the center O with respect to AB. The point Y traces out the sides AB, BC, and CA of the triangle ABC, while the point Z moves along the extensions of the sides BC, CA, and AB, respectively. Unlike in Experiment 1, Z moves along the ray starting at B, C, and A respectively, away from the triangle sides.

The set of points traced out by the point X, consists of three arcs of a circle, each of which has length $\frac{\pi}{3}R$ and lies on a circle centered at one of the vertices of ABC, of radius R. Here R is the radius of ABC's circumcircle. Using a dynamic geometry software, one can show that the starting point of each arc belongs to a circle containing one of the other arcs.

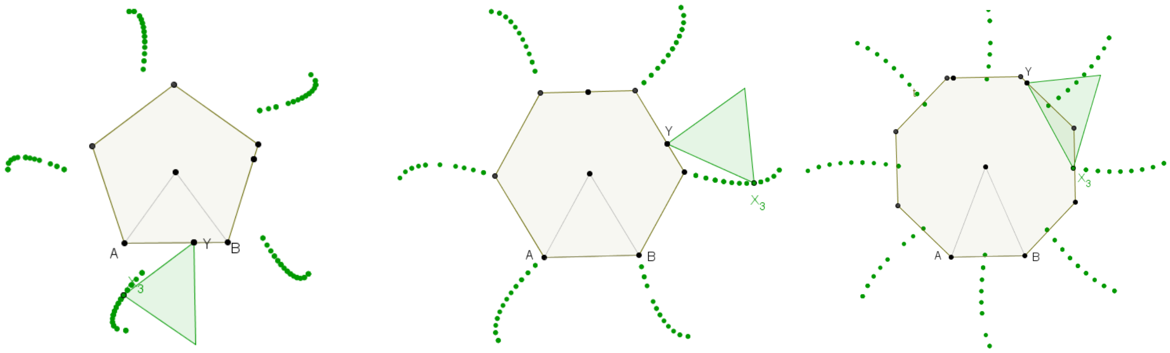




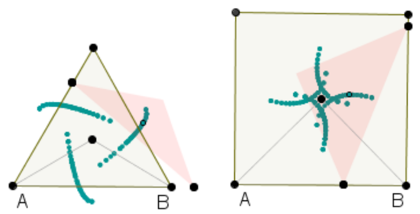
In the case N=4, the triangle XYZ starts its motion from initial position Y=A, Z=B, and X coincides with the point symmetric to the center O of the square ABCD with respect to AB. The motion of the points Y and Z is analogous to the motion described in the previous experiment.

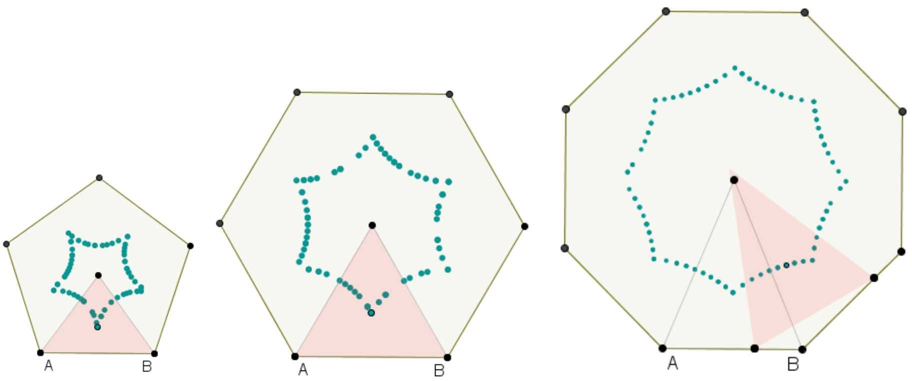
The set of points traced out by X consists of four segments of length $R(\sqrt{2}-1)$, lying on lines parallel to the square's diagonals. Moreover, these segments are traced out twice. The starting point of each segment belongs to a line containing another one of these segments. The point X is the farthest away from the center when the segment XY is perpendicular to the side containing Y. This is how one finds the length of the segment.

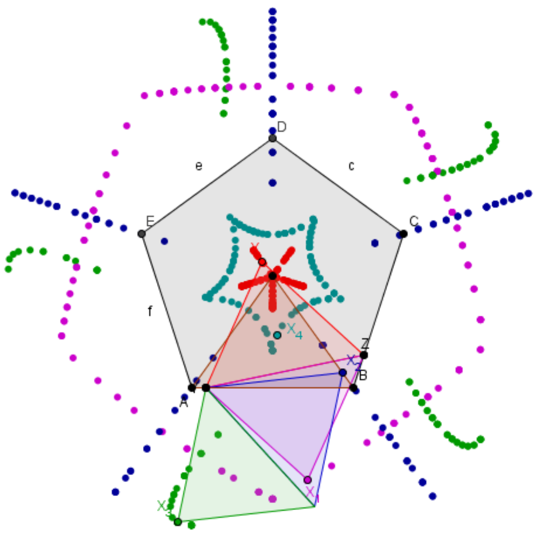
In the case N≥5, the set of points traced out by X consists of N curves.



Experiment 4. Let us consider the set of centroids of the triangle XYZ. In the case N≥4 the initial position of XYZ is Y=A, Z=B, and X=O and its motions is as described in the main problem. In the case N=3, the initial position of XYZ is Y=A, Z=C, X is symmetric to O with respect to AC. The point Y traces out the sides AB, BC, and CA of the triangle ABC, while the point Z moves along the continuations of the sides BC, CA and AB, respectively. Z moves along the ray staring at B, C, and A respectively, away from the triangle sides.







Thus an “old” problem explored first by programming in *Geomland*, and recently by means of *GeoGebra* [3] illustrates how to overcome some problems when teaching the notion of *locus*. While in the traditional classes my students have rarely been able to understand it in depth, observing the behavior of the moving point has enhanced both their understanding of loci-related problems and their self-confidence to explore further – to formulate new conjectures, to ask new questions and to enrich their knowledge in geometry.

Reference

[1] http://sunsite.univie.ac.at/elica/PGS/OVERVIEW.HTM

[2] Atanasova, S., Slavcheva, T. Applications of the *Geomland* for Solving *Loci Problems* Given on Exams, Sofia, 1988 (in Bulgarian)

[3] http://www.geogebra.org/cms/