Appendix III

When You Simply Decide to Dream…

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This is a story about mathematics, but not about the one in the textbooks which is so stale that it is almost identical to what you could find in a textbook from the 1960s. This is a story about the other one – the beautiful, limitless, full of life and art, as well as science; the one that allows you to dream, to experience wonderful moments, to feel the *dynamics*.

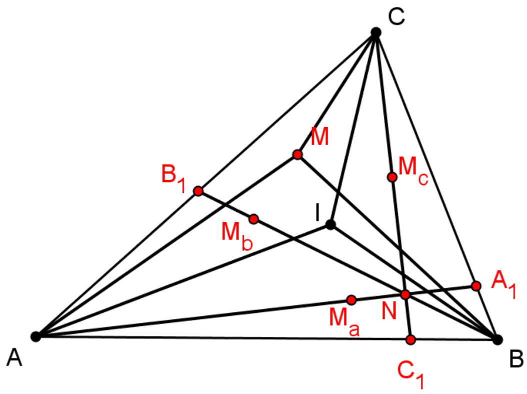
Geometry has always been and will continue to be the part of science which gives the best basis for defining infinity, precisely because in the geometric constructions and configurations this otherwise strong notion actually remains an elusive riddle. Questions like „What if…“ and „What if not…“ are the key to the limitless. And when we add some simple additional adornments like "But", "However", "Nevertheless", etc., to our geometric research, we arrive at utter chaos, the thoughts start jumping one over the other, the mathematician becomes a dreamer.

The story begins like this –two boys, participants in a summer school in mathematics, decide to "dive" into the infinity of a particular direction of geometry. The topic is related to the notion of *isogonal conjugates*, its applications, and its relation to famous geometric compositions.

The initial idea of our adviser, who posed the problem to us, was to synthesize widely known results in the area and unify them into a logically linked sequence. We did this, but it wasn't that easy to stop there. There were just a few essential theoretical facts and definitions, the main ones of which are listed below:

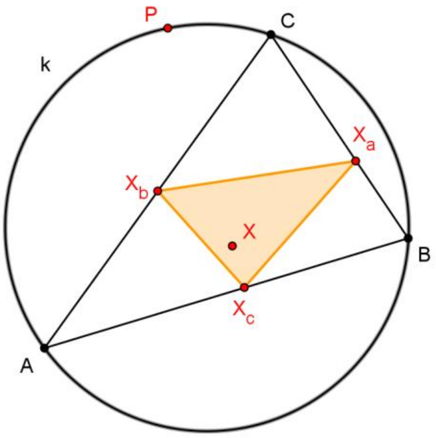
Definition 1

Given a $\triangle ABC$ and a point M in its plane, let M_a, M_b and M_c be the reflections of M with respect to the angle bisectors at A, B, and C respectively. Then AM_a, BM_b and CM_c are the ***isogonal conjugate lines*** *of AM, BM and CM respectively. They concur at a point N, called the* ***isogonal conjugate of point*** *M.*



Definition 2

For a ΔABC and an arbitrary point X in its plane, let Xa, Xb and Xc be the orthogonal projections of X onto BC, CA and AB. Then, the ΔXaXbXc is called the **pedal triangle** *of X for ΔABC.*



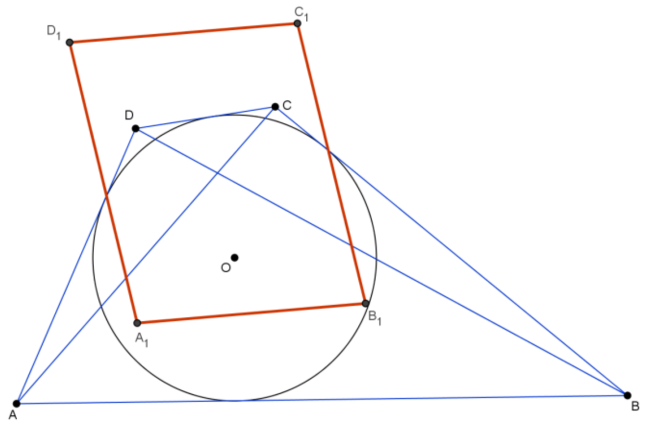
We explored relationships between various elements of the pedal triangles of two isogonally conjugate points. Studying the theory of isogonal conjugates "in depth" was interesting, but we felt we were lacking something in order to feel truly satisfied. Graphing and combining already known configurations and expressing their properties was exciting, but insufficient. We wanted *to find something new.*

After the summer school we followed our mentor's advice not to deal with too wide a theory, but rather concentrate on something concrete. But what could it be? Tens of mathematicians around the world had worked on discovering various properties of geometric constructions in this area and it would definitely not have been very effective for us to walk on the well trodden path. Therefore, we decided to look at the problem "from a different angle", to choose a different plane (as one would say if one were to use the stereometric language).

Above all, we need to acknowledge one of the most important elements of the work presented herein - the dynamic software. In this case the well known plane geometry software *GeoGebra* [1] was very appropriate to use. The contribution of *GeoGebra,* however, is certainly not restricted to the nicely graphed constructions, or the accessibility of the complex and crowded configurations. The main tool we used, which was very effective, was the fact that *GeoGebra* allows one to define functions combining a sequence of frequently repeated operations. We created a button which upon clicking it, would instantly create the isogonal conjugate of a given point for a given triangle. This is a simple construction, but it involves 7 different geometric constructions. For example, we were interested in constructing the isogonal conjugates of a fixed point with respect to 12 triangles. A simple calculation shows that it would have been terribly disheartening and ineffective to do this on a piece of paper, especially when everything is based on an idea or a hypothesis.

From our past research, we had substantial experience with dynamic studies of quadrilaterals circumscribed around a circle, conic sections (in particular ellipses), etc. –But what could we do by taking a quadrilateral circumscribed around a circle? Of course, it made sense to consider the triangles formed by the given points. Namely, we would form four triples of points using permutation. So far it was clear we would construct isogonally conjugate points with respect to these four triangles. But the question arises - conjugates of which point? Clearly the condition of the inscribed circle comes into play here. Of course, the first element associated with this incircle that comes to mind is its center...

We started in this way, almost as a joke, but the result was impressive. The four points formed a parallelogram!

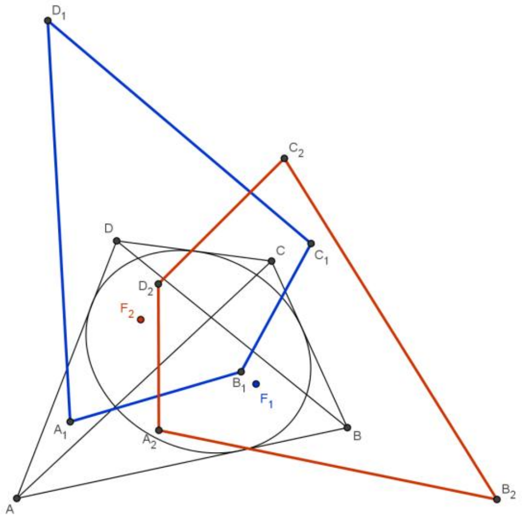


Conjecture 1

ABCD is a quadrilateral circumscribed about a circle centered at O. Let the points A_1, B_1, C_1, D_1 be the isogonally conjugate points of O with respect to ΔDAB, ΔABC, ΔBCD, ΔCDA, respectively. Then the quadrilateral $A_1B_1C_1D_1$ is a parallelogram.

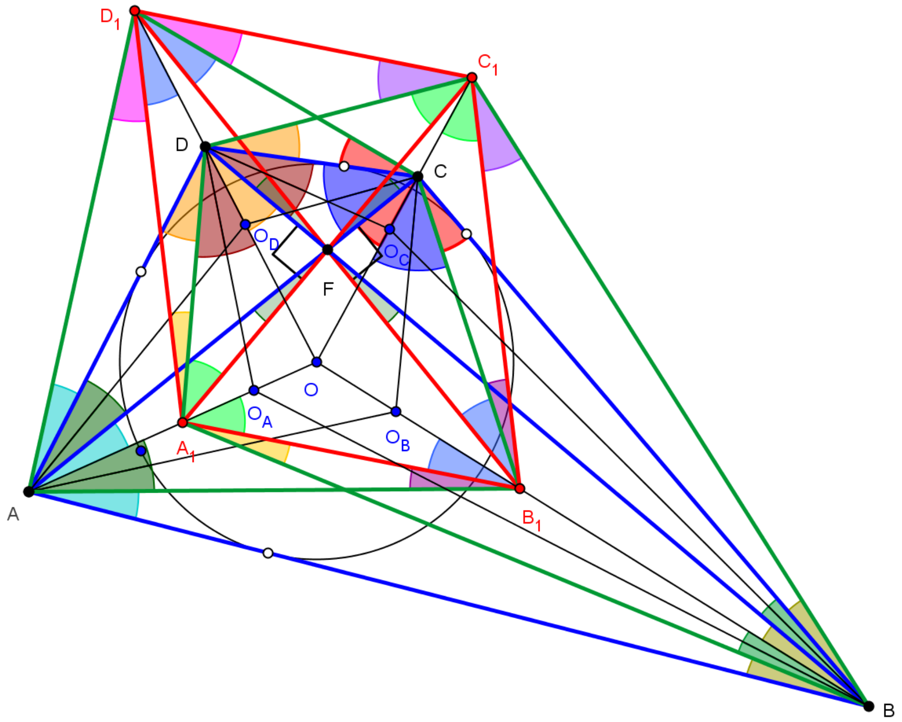
One would wonder why exactly a parallelogram is formed and what is the reason an arbitrary quadrilateral satisfying a single condition (that it is circumscribed about a circle), would yield such a special geometric figure after applying the "isogonal conjugacy" button a few times. The answer was also to appear in other similar questions. In fact, the reason for this "beautiful" result was the fact that under isogonal conjugacy, everything contains an underlying symmetry. Intrinsically connected to this problem are the particular cases of a rhombus, square, etc., for which one needs to impose additional conditions.

Certainly, one could enter the area of observations and hypotheses (and later their rigorous proofs) lead by meticulous logic. This is the way we tried to proceed. The next step was to get a generalized version of the already considered configuration. What could make more sense than substituting an ellipse for the circle and repeating the construction... However, in this case the center of the circle corresponds to the two foci. Then, we constructed the corresponding two quadrilaterals, using the algorithm described above. The result? What we obtained in this way were not parallelograms but two arbitrary quadrilaterals, which, however, were ... congruent!



Conjecture 2
Let $ABCD$ be a quadrilateral circumscribed about an ellipse with foci F_1 and F_2. Denote by A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2 the isogonally conjugate points of F_1 and F_2 with respect to ΔDAB, ΔABC, ΔBCD and ΔCDA. Then, the quadrilaterals $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are congruent**.**

This conjecture was shortly accompanied by a theoretical proof. It would probably make an impression on every reader not so much with the methods used, but rather, with the complex geometric constructions, that made us stare at them for days before we discovered what was in front of our eyes all along. These moments, however, are what draws one to do research in geometry.

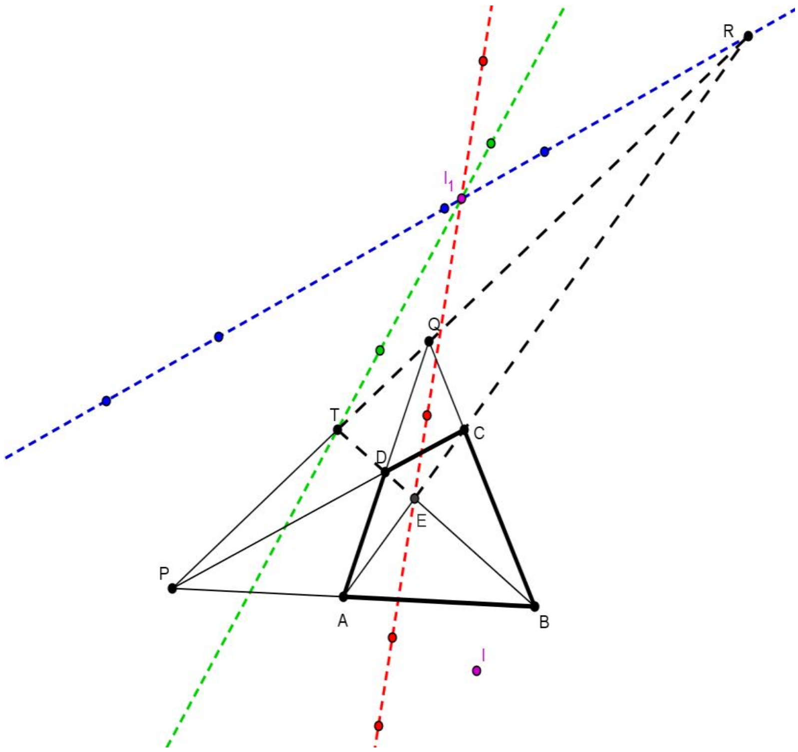


After establishing such interesting properties coming from applying "isogonal conjugacy" to two basic configurations, it occurred to us to slightly modify the initial conditions. The goal was to again consider a quadrilateral, but without imposing any conditions. The point to be conjugated was arbitrary, and the triangles with respect to which it was to be conjugated were formed by two vertices and the intersection point of the diagonals, rather than triples of vertices. This procedure would yield four triangles, and consequently, four isogonally conjugate points. It turned out that this did not yield a quadrilateral, simply because the points obtained (by applying the "isogonal conjugacy" button) were collinear. Moreover, it turned out that the line containing them passes through the intersection point of the diagonals...

Following this way of thinking, we were lead to consider an analogous construction based on the so called "complete quadrangle", which in fact has six vertices and three diagonals. These three diagonals yield respectively three intersection points, one of which we had already considered. We expected that the other two, given the proper choice of quadruples of vertices out of the six, would also yield two lines containing five points. Well, symmetry had a say in this as well. Everything turned out to be exactly the way we expected.

Conjecture 3

Let ABCDPQ be a complete quadrangle and let $AC \cap BD = E$, $AC \cap PQ = R$, $BD \cap PQ = T$ *be the intersection points of its diagonals. For an arbitrary point I in the plane denote by* E_1, E_2, E_3, E_4 *the isogonally conjugate points of I with respect to* $\triangle ABE$, $\triangle BCE$, $\triangle CDE$ *and* $\triangle DAE$. *Let* R_1, R_2, R_3, R_4 *be the isogonally conjugate points of I with respect to* $\triangle APR$, $\triangle PCR$, $\triangle CQR$ *and* $\triangle QAR$, *and* T_1, T_2, T_3, T_4 *be the isogonally conjugate points of I with respect to* $\triangle BPT$, $\triangle PDT$, $\triangle DQT$ *and* $\triangle QBT$. *Then the points* (E, E_1, E_2, E_3, E_4), (R, R_1, R_2, R_3, R_4) *and* (T, T_1, T_2, T_3, T_4) *are collinear and these three lines concur at a point isogonally conjugate to I with respect to* $\triangle ERT$.



We didn't expect one thing, though, the final part of the theorem, possibly the strongest result about this construction. The essential thing here is that the three lines determined above intersect at one point. Moreover, the intersection point is not an arbitrary one. By simply constructing its isogonally conjugate point with respect to the triangle with vertices the three intersection points of the diagonals, we arrived at a conclusion exceeding our wildest expectations. The constructed point turned out to be the point we started at – the arbitrarily chosen and then fixed point in the plane of ΔABC…

This was not difficult to prove, but it was clear that the reasoning in the opposite direction would have been possible only given great luck.

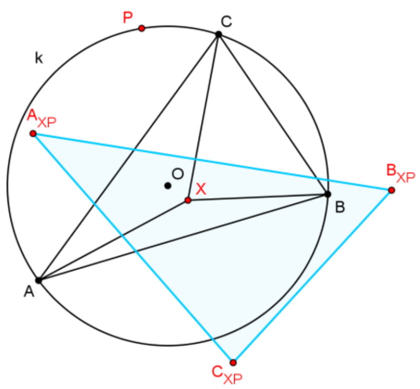
Starting from a simple configuration, we developed the idea by changing some of the parameters and concluded by arriving at the closure of a frame, which lead us to believe that this type of improvisation has been exhausted. However, based on these achievements, we knew we had to continue ahead.

At one point our adviser noted something which was bound to lay the foundation of the biggest results in the research that followed. His note was both simple and elegant due to the sound logical argument it was based on. It was true we had derived wonderful properties using quadrilaterals, but why did we neglect to consider configurations based on triangles?...

Many experiments followed, as well as various kinds of substitutions and other approaches, until we finally figured out the fundamental construction. It was... a triangle and a point lying on its circumcircle. Clearly, we were to construct points isogonally conjugate to this point with respect to certain triangles. All that was left to do was to determine the triangles with respect to which we were to apply the isogonal conjugacy, analogously to the approach taken in our previous studies. We needed one more point X, which together with the three permutation pairs of the fundamental triangle would form the needed configuration – a point and three triangles. Naturally, before figuring out a more general dependency under less restricting conditions, we decided to choose X to coincide with the center of the circumcircle...

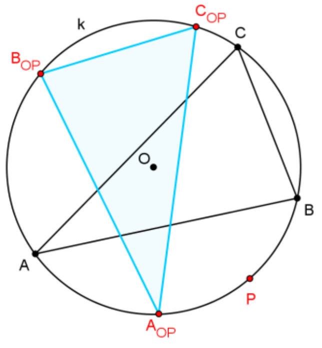
Definition 3

Let P be a point on the circumscribed circle of a ΔABC and X be a point in its plane. An isogonal triangle of P, centered at X, is the triangle determined by the isogonal conjugates of P with respect to the triangles ABX, BCX, and ACX.



And then came the moment when our substantial literature review, which filled our heads with thousands of properties and relations, yielded a relation which subsequently directed us in our search for a result. In many places we encountered the proposition that for a given triangle and a point in its plane, the pedal and the Cevian triangles are similar. During the Summer school of the High School Institute of Mathematics and Informatics (HSSI) [2] we studied the properties of pedal triangles of isogonally conjugate points anyway, so we decided that it makes sense to include this component in our research. It turned out we were right.

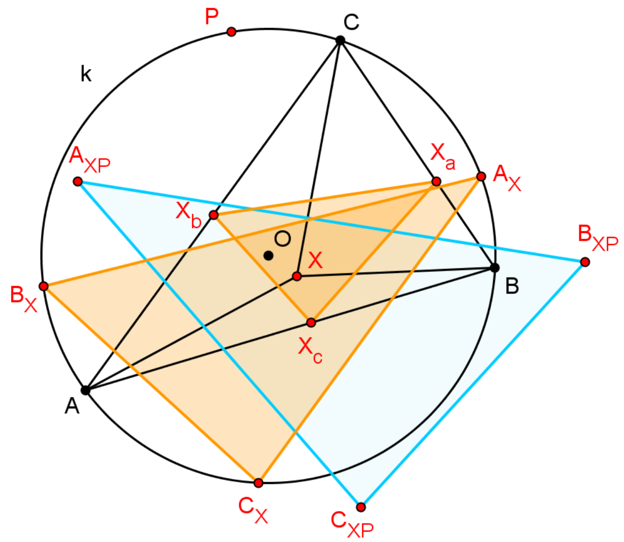
What came out of this? A corollary to a lemma proven by us shows that the three new points, which we later came to call "isogonal triangle" lie on the circle. Moreover, the new triangle was similar to the initial triangle! Due to the fact that they have the same circumcircle, it turned out they are in fact congruent! We didn't feel this was a coincidence, rather, it was clear to us that this particular case somewhat enhances the "extraordinary nature" of the result obtained.



We needed to substitute the center of the circumcircle by a different point. We chose the center of the incircle as an analogue in this case. This time there was no congruence, and only part of the previously existing similarity remained. In fact, we hadn't thought of the fact that the Cevian triangle (and respectively the pedal triangle) of the circumcenter of a given triangle, are in fact similar to this triangle, and therefore are also similar to the isogonal triangle in this particular case. It turned out, in fact, that the latter was true not only for this case. We were now ready to use the software's capabilities to note the following significant property. Given an arbitrary point *X* in the plane of a given triangle, its isogonal triangle with respect to an arbitrary point on the circumcircle is similar to the pedal triangle, and respectively, to the Cevian triangle with respect to *X*. At last, we had taken the crucial step – we had a stand-alone proposition. We initially called it Main Hypothesis, and it eventually turned into the following:

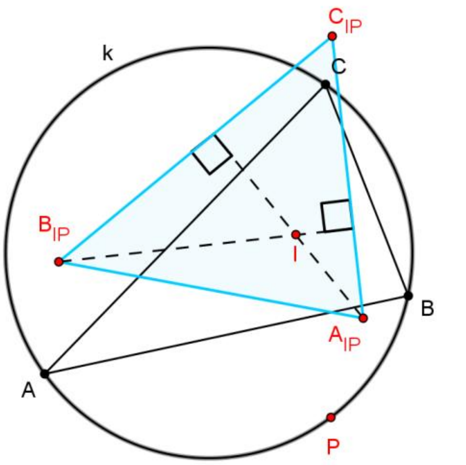
Main Theorem:

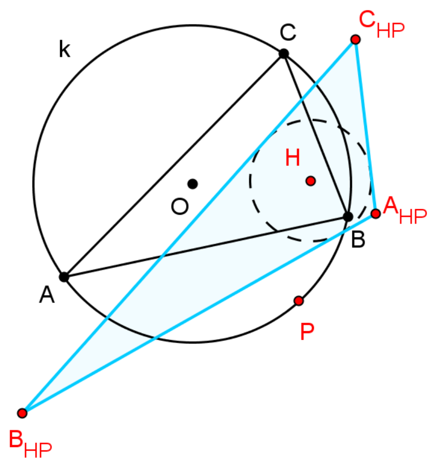
Let X be a point in the plane of a $\triangle ABC$. Then for every point P on the circumcircle of $\triangle ABC$, the isogonal triangle of P centered at X is similar to the Cevian and the pedal triangles of the point X with respect to the $\triangle ABC$.



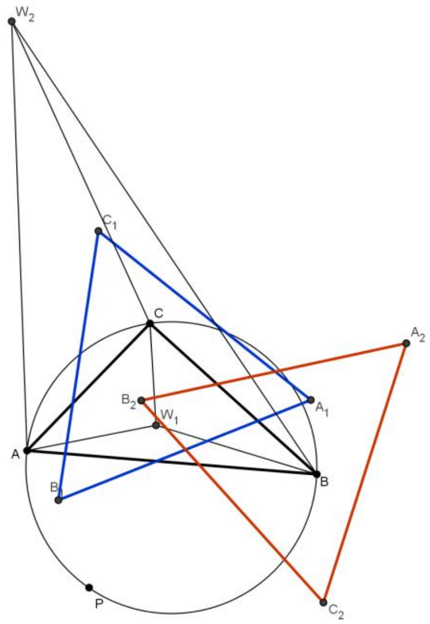
It turned out that the proof wasn't easy at all. Breaking the process apart into small steps that followed the logical evolution from particular cases to the general theorem allowed us to gradually develop small ideas, expressed in four lemmas and corollaries. The conclusion came in the end – using dilation we were able to finish the proof of the theorem. Not only this, but the dilation implies that the locus of the vertices of the isogonal triangles consists of three circles respectively! That is how we concluded that the initially chosen point on the circumcircle was auxiliary, while the point X was the one which determined everything. This is a good example of how sometimes a partial mistake can lead to a truly valuable result…

We managed to prove a number of additional properties considering some special cases, e.g. when X coincides with the center of the circimcircle and then with the center of the incircle. Further on, we found special relations involving the first and second Brocard points.





Here we decided to stray aside slightly and to consider the two *isodynamic* points of the reference triangle (the intersection points of the three Apollonius circles for the triangle's vertices).

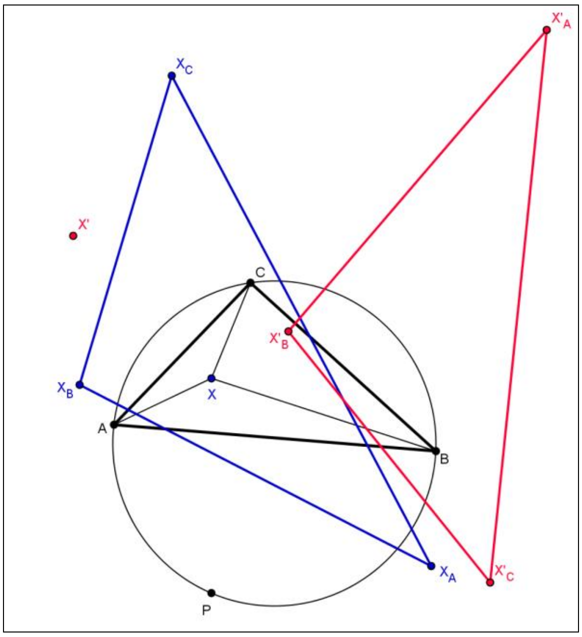


We constructed their isogonal triangles and according to our theorem, they were similar not only to each other, but also to the pedal and Cevian triangles for these two points. Thus we obtained two families of congruent equilateral triangles.

The proof was based on dilation again. We noticed that the congruency of all triangles holds true in the general case, i.e. can formulate the following:

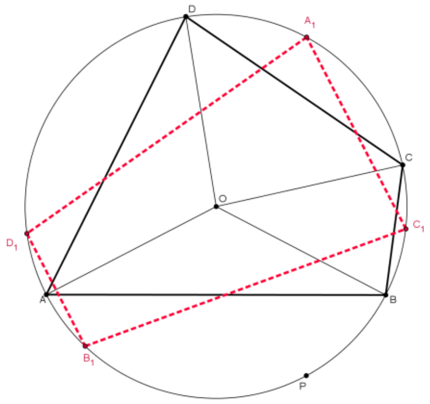
Conjecture 3

All isogonal triangles of two inverse points with respect to ABC's circumcircle, are congruent.



Following the logical generalizations, we came back to constructions involving quadrilaterals. Analogously to what we did with the isogonal triangles we decided to consider various configurations. We arrived at an interesting result only for the case when the point with respect to which isogonal conjugates were taken coincided with the circumcircle of the initial quadrilateral.

If the quadrilateral were inscribed in a circle family of isogonal quadrilaterals consisted of quadrilaterals inscribed in the same circle. Moreover, they were trapezoids. The proof was not difficult, despite the unusual result, given the arbitrariness of the original quadrilateral.



In conclusion, this project involved substantial research leading to numerous results relating innovative constructions with well known geometric configurations. By no means have we exhausted the opportunities for research along these lines. What is important is to simply decide to dream...

Acknowledgements

We would like to acknowledge the valuable ideas, advice and critics of our mentor, Mr. Dimitar Belev, who supported us in all difficult moments. A special gratitude is due to Prof. Georgi Ganchev and Prof. Oleg Mushkarov from the Institute of Mathematics and Informatics who followed the development of our project and suggested constructive ideas for its improvement. We would also like to thank Acad. Petar Kenderov who saw more than just two high school students at our presentation of the project; he saw us as young researchers - something we have always wanted to be! Last but not least, we are very grateful to our teachers from the *National High School of Mathematics and Natural Sciences "Academician Lyubomir Chakalov"*, Sofia, Bulgaria, for having encouraged our work on this project for more than two years.

Reference

[1] *GeoGebra* http://www.geogebra.org/cms/

[2] *High School Institute of Mathematics and Informatics* http://www.math.bas.bg/hssi/