Fractals – broken with no need for repairs

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1 Introduction

For centuries, we tried to describe nature with simple geometric forms: Circles, squares, cones, cylinders … In other words, objects from what we call Euclidian geometry. In the 1970s and 1980s Benoît Mandelbrot challenged that view with his famous proverb “Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line!” To explain this, he used several models, of which the length of a coastline is probably the most well-known: A satellite photography of the UK coastline looks like a soft, uninterrupted line that is easy to measure. Flying on an airplane over the same coast, one can already see jagged places, indentations, etc. by which the coastline appears much longer. If one changes to a small hang-glider, one can see beaches, riffs, harbours, etc. making the coastline looking even longer than from the plane. And if one walks along the coast, the line looks even more jagged and bizarre – and again, longer! We can see where this is going – using finer and finer resolution, the coastline gets longer and longer. In fact, it is infinitely long! Yet, the area of the UK is not infinitely large – but how can you have a figure with a finite area, yet an infinite circumference? With the above-mentioned Euclidian forms, you cannot; but with fractal forms, you can! Let’s see how this works!

2 What are fractals?

2.1 Defining the formless forms

As you can imagine, it is not quite easy to define something that even Euklid called “formless” (Mandelbrot himself coined the term *fractal form* or simply *fractal*, from the Latin word *fractus*, meaning fractured or broken). There are several possible definitions of fractals, of which we will give the one that best sums up the properties of fractals.

Definition: A structure (e.g. an object or a set) is called *fractal*, if it shows self-similarity and if it has a non-integer dimension.

To understand this definition, we need to discuss two things – self-similarity and dimension.

2.2 Self-similar structures

What do we mean with self-similarity? A typical example in nature would be a big tree. If you take a major branch of it, it basically looks the same as the whole tree, only smaller. If you break off a smaller branch from the big one, it still looks pretty much similar to the big branch (and to the whole tree), but again smaller. If you break off a small side branch from this, it again looks similar, but smaller. In nature, you can only do this so often, but in fractal geometry, you can repeat this over and over again and will always end up with an object that looks similar to the whole. So we can define:

Definition: An object is *self-similar* if parts of it contain shrunk (not necessarily exact) copies of itself. An object is *exactly self-similar* if it consists only of exact copies of itself.

Before this gets too confusing, we will show an example of an (exactly) self-similar object. It is called the Koch-curve (we will see how to construct this one below).

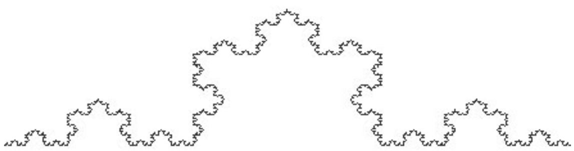


Fig.1 Koch-curve

You might not see it immediately, but this curve consists of four copies of itself. To show this better, we will colour the four parts in different colours:

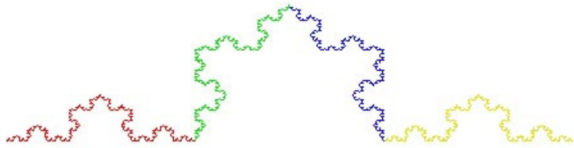


Fig.2 Koch-curve in colours

Now the red part of the curve looks exactly the same as the whole curve, only smaller. If we would take only the left part of the red curve, it again would look the same as the whole curve.

2.3 Broken dimensions

We are used to integer dimensions: Lines have dimension 1, planes, squares, rectangles, circles etc. have dimension 2, cubes, spheres, cones etc. have dimension 3. But what have we learned in the first paragraph? “Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line!” So obviously, to describe fractals, we need dimensions that are not integer numbers. For instance, the Koch-curve has dimension 1.262, which means it is something between a line and a plane. The closer the dimension is to 2, the more the fractal “fills” the plane.

OK, we need non-integer dimensions, but how do we define them? There are several possibilities to do that. We will use the so-called *Hausdorff-Dimension* (as Mandelbrot did), and actually only a special case of it, the so-called *self-similarity dimension* (which works for exactly self-similar objects).

Definition: A geometric object consisting of n disjunctive parts that are exact $1:m$ copies of this object has the *self-similarity dimension*

$$D = \frac{\log n}{\log m}$$

It is fairly easy now to determine the dimension of the Koch-curve. It consists of four copies with a scale of 1:3, i.e. the dimension is

$$D = \frac{\log 4}{\log 3} \approx 1.262$$

Nicely enough, the “usual” (i.e. Euclidian) dimension is also included in the self-similarity dimension, which means if one calculates the self-similarity dimension of, say, a square, one would still get dimension 2. Why is that? Well, let’s consider a square:

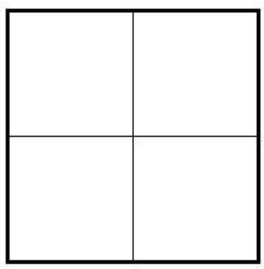


Fig.3 Square in four parts

As we can easily see, the large square consists of 4 small squares, which are just 1:2 copies of the large square. The self-similarity dimension would hence be:

$$D = \frac{\log 4}{\log 2} = \frac{\log 2^2}{\log 2} = \frac{2 \cdot \log 2}{\log 2} = 2$$

Tasks:

[1] Make a similar proof to show that the self-similarity dimension of a rectangle is also 2.

[2] Make a similar proof to show that the self-similarity dimension of a cube is 3.

3 How to make fractals

Until now, we have just analysed existing fractals. But how do you actually construct a fractal? And why, if the calculations are so easy, has no one really done this until the late 20th century?

The first question – how to construct a fractal – has several possible answers. The reason for this is that the term fractal, by its definition, is very broad, i.e. a lot of pretty different structures are of a fractal nature, and there is no way of having just one construction method for all of them. But there is a large class of fractals that are all constructed the same way.

3.1 Classic fractals

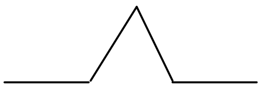
With the term *classic fractal* we mean a fractal that is exactly self-similar and can be constructed by means of an iterative process of replacement: a start object (mostly a line or segment), the so-called *initiator*, is replaced by another geometric figure (consisting of several initiators), the *generator*. In the resulting figure each initiator is again replaced by a generator, and this process is repeated indefinitely. The fractal is the limit of this iteration. Mathematically we can describe this as a recursive function or, for that matter, as a sequence:

$$x_{n+1} = f(x_n),$$

where x_0 is the initiator and $x_1 = f(x_0)$ is the generator.

Before this gets too confusing, we will again demonstrate it with the help of the Koch-curve.

Example: The initiator of the Koch-curve is a line segment. The generator consists of four segments whose length is one third of the original segments' length, given by trisecting the original segment and replacing the central third of the segment by an equilateral triangle whose baseline is cut out. Writing this as a recursive function makes it clearer:

$x_0 =$ ——— $x_1 = f(x_0) =$ 

These four segments are then again replaced by the generator, resulting in a structure of 4x4=16 segments. These segments are yet again replaced by the generator etc. This results in the following construction:

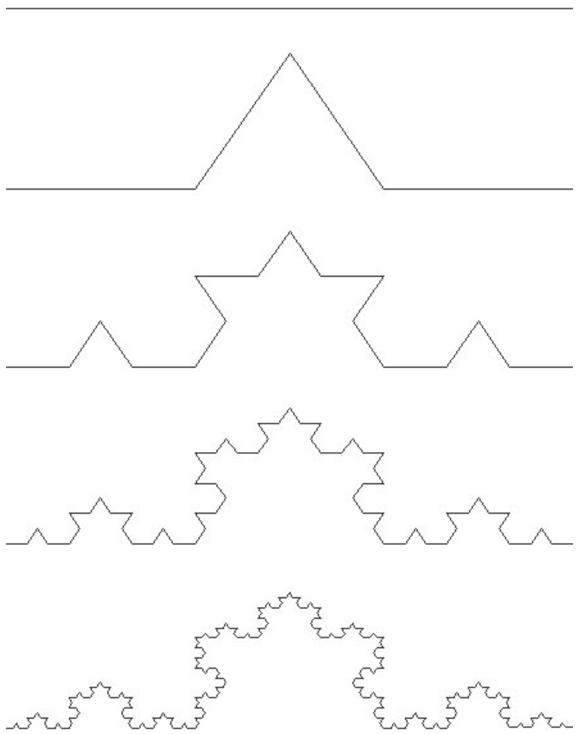


Fig.4 Iterations of the Koch-curve

If we would continue indefinitely, we would get the Koch-curve. But even the first five iterations already show a very good approximation of the curve. The construction also answers the second question that we raised above: Why has this not been done earlier in history? Well, some constructions have been done earlier, but if you try to make this construction by hand, you will soon find out that even making the third or fourth iteration is already a very tedious process. Only with the help of computers was it possible to make constructions like these in a reasonable time and with the necessary accuracy.

3.2 Can I do this, too?

Absolutely! All you need is some programming language that you are familiar with. We have prepared some constructions using Logo, but you may as well use any other programming environment (e.g. a Java version; Logo versions of numerous fractals can be found in [1]). It is also easy to change the programs to create your own fractals – the only thing that you would need to change is the recursive function. Let's first have a look at the recursive function of the Koch-curve program:

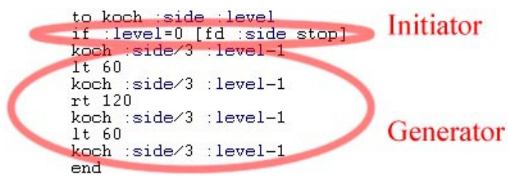
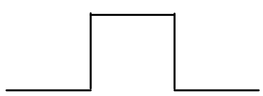


Fig.5 Recursive function in program *koch_curve.lgo*

Even if you are not a programmer, you can see what's going on here: The initiator draws a line segment with a given length in a given angle, and the generator consists of four line segments (whose length is one third of the original line segment) that go first straight, then in a 60° angle left, then 120° right, and again 60° left to bring it back to a straight line, hence creating a triangle (with no baseline) as depicted above. Now let's assume we want to change the generator to not have a triangle in the middle, but a square, i.e. have something like this:

$x_0 =$ $x_1 = f(x_0) =$ 

This means that the generator consists of five lines that go first straight, then in a 90° angle left, 90° right, again 90° right, and 90° left to be back at a straight line. In the program, that would look like this:

```
to koch :side :level
if :level=0 [fd :side stop]
koch :side/3 :level-1
lt 90
koch :side/3 :level-1
rt 90
koch :side/3 :level-1
rt 90
koch :side/3 :level-1
lt 90
koch :side/3 :level-1
end
```

Fig.6 Recursive function in program *koch_curve_square.lgo*

If we let the program run with this recursive function, we get the following:

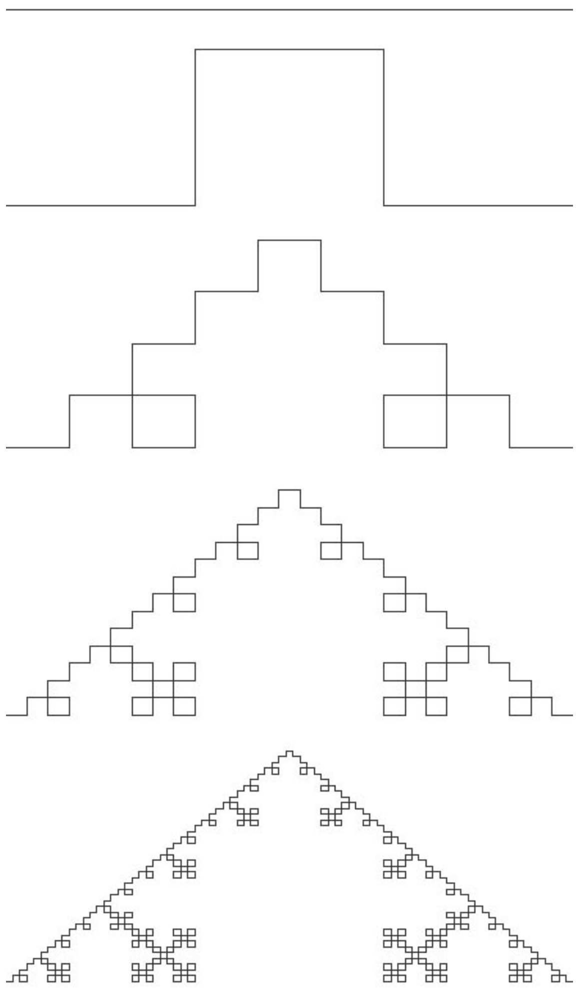


Fig.7 Modified Koch-curve with square generator

Task:

[3] Think of other geometric figures that may replace the triangle/square and modify the program accordingly.

4 Drawing endless lines

How long is the Koch-curve? To answer that question we recall that we can describe each classic fractal by a sequence, as described in 3.1 above. Also the length of a classic fractal can be described as a sequence. In our case (assuming that the initiator has length 1), we can calculate the length l_n of the n-th iteration as follows.

$$l_0 = 1,$$

so much is clear. We said above that "The generator consists of four segments whose length is one third of the original segments' length", i.e. the length of the first iteration would be

$$l_1 = 4 \cdot \frac{l_0}{3} = \frac{4}{3} \cdot l_0 = \frac{4}{3}.$$

In the next iteration, each one of the four segments is now again replaced by four segments whose length is one third of the former segment, i.e. its total length would be

$$l_2 = 4 \cdot \frac{l_1}{3} = \frac{4}{3} \cdot l_1 = \left(\frac{4}{3}\right)^2 \cdot l_0 = \left(\frac{4}{3}\right)^2.$$

We can see where this leads to. The n-th iteration has length

$$l_n = \left(\frac{4}{3}\right)^n \cdot l_0 = \left(\frac{4}{3}\right)^n.$$

As the Koch-curve is the limit of the sequence, the length of the Koch-curve is

$$l = \lim_{n\to\infty} l_n = \lim_{n\to\infty} \left(\frac{4}{3}\right)^n = \infty,$$

which means the Koch-curve is infinitely long (which also means that no one can really draw a Koch-curve, only an approximation)! Now how about the area beneath the curve? One would expect it to be also infinitely large. With classic Euclidian geometric objects, this would be the case, but how about fractals?

The initiator is just a straight line, i.e. the area beneath it is

$$A_0 = 0.$$

In the first iteration, this area is increased by the area of an equilateral triangle with side length $a_1 = \frac{1}{3}$.

Recalling the equation for the area of an equilateral triangle, we get

$$A_1 = A_0 + \frac{\sqrt{3}}{4} \cdot a_1^{\ 2} = \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3}\right)^2 = \frac{\sqrt{3}}{4} \cdot \frac{1}{9}$$

In the second iteration, this area is again increased by the areas of 4 equilateral triangles with side length $a_2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$, i.e. the total area is

$$A_2 = A_1 + 4 \cdot \frac{\sqrt{3}}{4} \cdot a_2^{\ 2} = A_1 + 4 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{9}\right)^2 = \frac{\sqrt{3}}{4} \cdot \frac{1}{9} \cdot \left[1 + 4 \cdot \frac{1}{9}\right].$$

In the third iteration, the area is again increased, this time by the areas of $4^2 = 16$ equilateral triangles with side length $a_3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$, which results in a total area of

$$A_3 = A_2 + 4^2 \cdot \frac{\sqrt{3}}{4} \cdot a_2^{\ 2} = A_2 + 4^2 \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{27}\right)^2 = \frac{\sqrt{3}}{4} \cdot \frac{1}{9} \cdot \left[1 + 4 \cdot \frac{1}{9} + 4^2 \cdot \left(\frac{1}{9}\right)^2\right]$$

Generally, the area beneath the n-th iteration would be

$$A_n = \frac{\sqrt{3}}{4} \cdot \frac{1}{9} \cdot \left[1 + 4 \cdot \frac{1}{9} + 4^2 \cdot \left(\frac{1}{9}\right)^2 + \cdots + 4^{n-1} \cdot \left(\frac{1}{9}\right)^{n-1}\right] = \frac{\sqrt{3}}{4} \cdot \frac{1}{9} \cdot \sum_{i=1}^{n} 4^{i-1} \left(\frac{1}{9}\right)^{i-1} = \frac{\sqrt{3}}{4} \cdot \frac{1}{9} \cdot \sum_{i=1}^{n} \left(\frac{4}{9}\right)^{i-1}$$

As the Koch-curve is the limit of the sequence, the area beneath the Koch-curve is

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{\sqrt{3}}{4}\cdot\frac{1}{9}\cdot\sum_{i=1}^{n}\left(\frac{4}{9}\right)^{i-1} = \frac{\sqrt{3}}{4}\cdot\frac{1}{9}\cdot\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{4}{9}\right)^{i-1} = \frac{\sqrt{3}}{4}\cdot\frac{1}{9}\cdot\sum_{i=1}^{\infty}\left(\frac{4}{9}\right)^{i-1},$$

To calculate this limit, we have to basically calculate the limit of a series, namely

$$s = \sum_{i=1}^{\infty}\left(\frac{4}{9}\right)^{i-1} = \sum_{i=0}^{\infty}\left(\frac{4}{9}\right)^{i}.$$

This is a geometric series with start element $b_1 = 1$ and ratio $q = \frac{4}{9}$. The sum of this series is

$$s = \frac{a_1}{1-q} = \frac{1}{1-\frac{4}{9}} = \frac{9}{5}.$$

Now we can easily calculate the area beneath the Koch-curve as

$$A = \frac{\sqrt{3}}{4}\cdot\frac{1}{9}\cdot\sum_{i=1}^{\infty}\left(\frac{4}{9}\right)^{i-1} = \frac{\sqrt{3}}{4}\cdot\frac{1}{9}\cdot s = \frac{\sqrt{3}}{4}\cdot\frac{1}{9}\cdot\frac{9}{5} = \frac{\sqrt{3}}{20}.$$

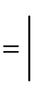
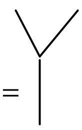
In any case, the area beneath the Koch-curve is finite, while its length is infinite!

Task:

[4] Calculate the length and area of the modified Koch-curve with square generator that we constructed above.

5 Growing trees

To see how close you come to nature with fractals, we have also prepared a construction of a tree in Logo (and in Java). This is a tree that starts out as just a trunk. After one year, the trunk grows two branches. The next year, each of the branches grows another two branches etc. Mathematically speaking, the initiator is a vertical line segment, the generator adds two line segments (with length ¾ of the original segment), one in a 30° angle to the left, one in a 45° angle to the right. In functional terms, this means

$x_0 =$  $x_1 = f(x_0) =$ 

Task:

[5] Calculate the length of the last branch of the fourth iteration of this tree.

[6] If you want to climb from the ground to the very end of one branch, what is the total length of your climb?

Here is how this tree looks like in its seventh iteration:

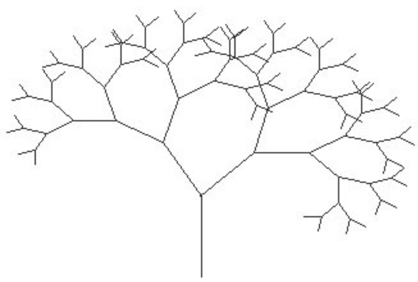


Fig.8 Tree in its seventh iteration

Task:

[7] Modify the software so that the tree grows three branches per year. Choose suitable angles and lengths for the branches (they may also have different lengths).

[8] How long is one branch in your tree's fourth iteration?

6 Fractals are everywhere

This chapter shows a few examples of more fractals, as well as where fractals and chaos theory (the underlying mathematical theory) is used also in everyday life.

6.1 Weather frogs or weather fractals?

"Weather chaos!" You may have read that headline somewhere, but did you know that there is some mathematical truth in it? Weather is what mathematicians call a *chaotic system*, which means that very small changes in conditions can have huge effects after a while. The same goes for fractals, where a small change in conditions can also have a large effect on the outcome. If you see how the original Koch-curve looks as compared to the modified Koch-curve, you can see that the two curves don't look alike at all, although we have only replaced one triangle by a square. This is what makes weather forecasts so difficult, particularly long-term predictions!

6.2 Julia and Mandelbrot sets

One of the very first dynamic systems that chaos theory researched sounds rather simple: "multiply a value x with itself and add a parameter c" (mathematically this is a sequence $z_{n+1} = z_n^2 + c$). If we start with $z_0 = 0.5$ and $c = 0$, we get the following sequence: 0.5, 0.25, 0.0625, 0.00390625 … After a few iterations, we can see where this is going – the result gets closer and closer to 0. Not very interesting, one would think. Yet different start values and parameters will lead to surprising results:

Task:

[9] Use a calculator or a computer to calculate the first couple of values for the above sequence with parameters $z_0 = 0.5$ and $c = -1$, and then with parameters $z_0 = 0.5$ and $c = 2$

One would expect behaviour similar to the one of the first sequence, but that does not happen at all! One set of values leads to an oscillation between the values 0 and –1, the other one does not seem to have any predictability whatsoever. Gaston Julia was interested in exactly this sequence and wanted to know for which (in his case complex) numbers the sequence converges or at least was bounded, and for which parameters it was unlimited. The filled Julia set (here is a speedier, but less instructive Java version) with a given parameter c answers this question: A complex number z_0 is an element of the filled Julia set, when the sequence $z_{n+1} = z_n^2 + c$ with starting element z_0 is bounded (limited), otherwise the number z_0 is not in the filled Julia set. If we color those elements of the complex plane in black which are elements of the filled Julia set, we get the following (here we used parameter $c = -1$):

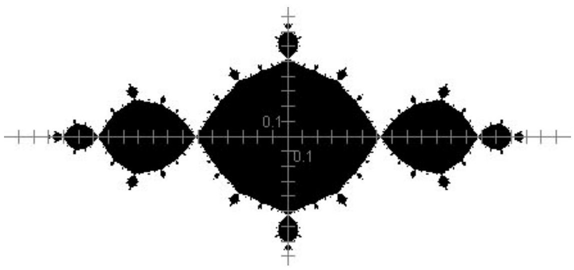


Fig.9 Filled Julia-set with $c = -1$

If we zoom into the set, we can observe self-similarity again:

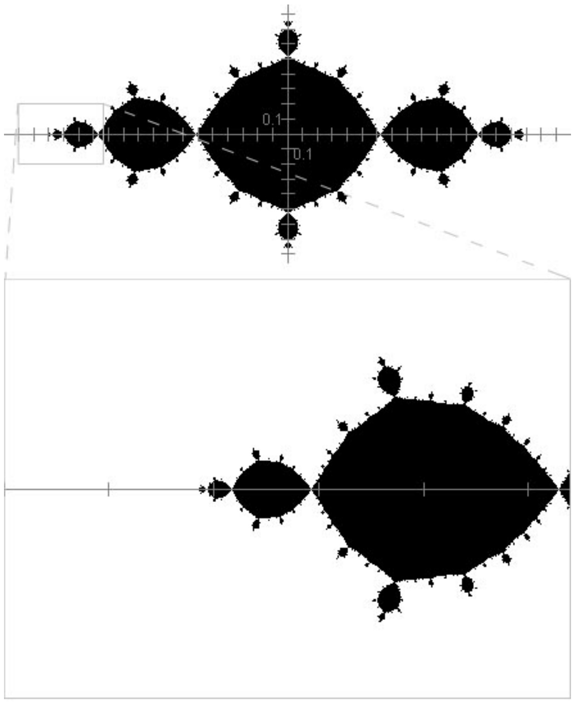


Fig.10 This looks very (self-) similar...

Task:

[10] Draw the filled Julia sets with $c = -1.25$ and then with $c = -1.4$. What can you observe?

[11] Draw the filled Julia set with $c = -0.5 + i \cdot 0.6$.

These tasks show clearly that even small variations in parameters lead to very different outcomes. An overview of results for different values of *c* can be found in [2].

Benoît Mandelbrot, who was a student of Gaston Julia, also was interested in the sequence $z_{n+1} = z_n^{\ 2} + c$. While Julia left the parameter *c* fixed and checked the sequences behavior for different starting values z_0, Mandelbrot did it the other way around: He wanted to know the behavior of the sequence for different values of c, always starting with $z_0 = 0 + 0i$. The Mandelbrot set (Logo or Java version) is the result of his work: A complex number *c* is an element of the Mandelbrot set, when the sequence $z_{n+1} = z_n^{\ 2} + c$ with starting element $z_0 = 0 + 0i$ is bounded (limited), otherwise the number *c* is not in the Mandelbrot set. If we color those elements of the complex plane in black which are elements of the Mandelbrot set, we get the following famous representation, which is also known as the “apple man”:

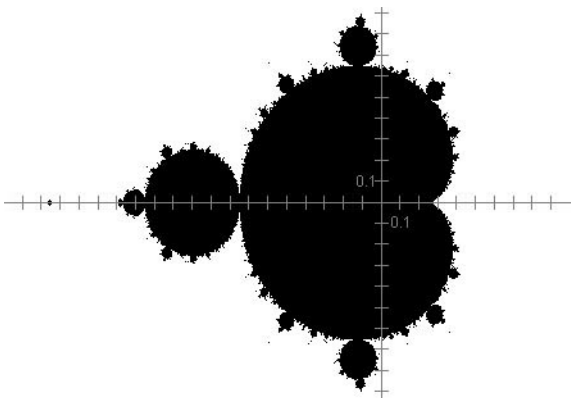


Fig.11 Mandelbrot set

Task:

[12] Zoom into the Mandelbrot set (particularly the border regions are interesting) and see whether you can discover self-similarity.

Sometimes one can see more colourful representations of the Mandelbrot set (Logo or Java version). This can be done by not only distinguishing between black (the sequence is bounded) and white (the sequence is not bounded), but assigning different colours to different “speeds” of divergence. This colour assignment can happen in various ways, so the following picture is just one of many representations:

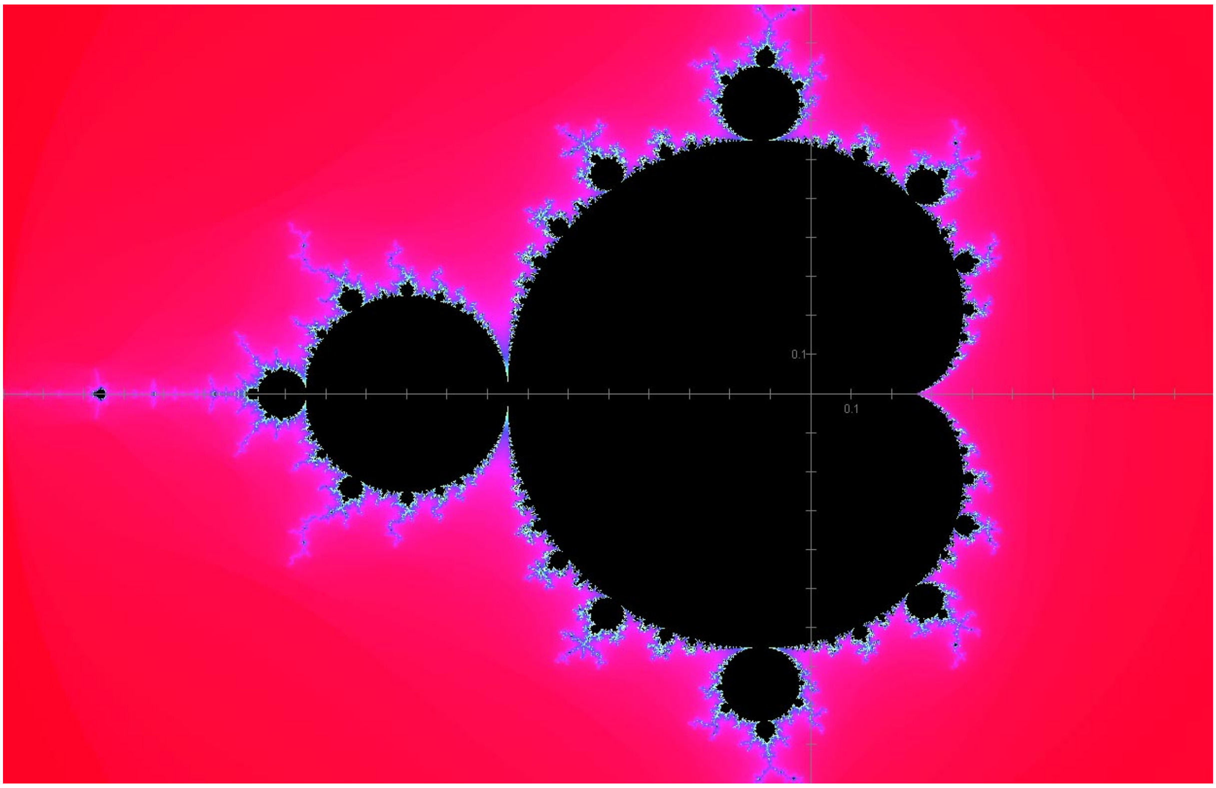


Fig.12 Colourful apple man

The pictures look very nice, but how are they actually created? Both the Julia and the Mandelbrot set would theoretically require calculating the limits of a lot of series. Not only would this be mathematically difficult, it would require very complex software – and a long time. But we can create very good *approximations* of the Julia and Mandelbrot sets fairly easily, because we actually do not need to know the exact *value* of the limit, we only want to know whether the sequence *has* a (finite) limit. For this, it can be shown that if the sequence $z_{n+1} = z_n^2 + c$ (with either a fixed parameter c and a variable starting element z_0 in the case of Julia sets, or with a variable parameter c and a fixed starting element $z_0 = 0 + 0i$ in the case of the Mandelbrot set) reaches a value z_i with $|z_i| > 2$, then the sequence is not bounded, i.e. has no finite limit. For the exact calculation of the Julia or Mandelbrot sets, this would not help us all too much, because how would we find out whether the sequence reaches a value z_i with $|z_i| > 2$ for some index i? But for an approximation this is very helpful, because we can simply say the following: Let’s calculate the first few values of z_i, and if they fulfill $|z_i| \le 2$, we can be fairly sure that the sequence remains bounded. To be a bit more exact, we choose a number m (a typical value would be e.g. $m = 20$, then calculate $z_0, z_1, z_2, \cdots, z_{25}$, and if all these values fulfill $|z_i| \le 2$, then we say the sequence is bounded. If one of the values fulfils $|z_i| > 2$ then we stop and say the sequence is not bounded. The higher the value for m is, the better the approximation. To demonstrate this, we can have a look at the approximation of the filled Julia-set with $c = -1$ for different values of m:

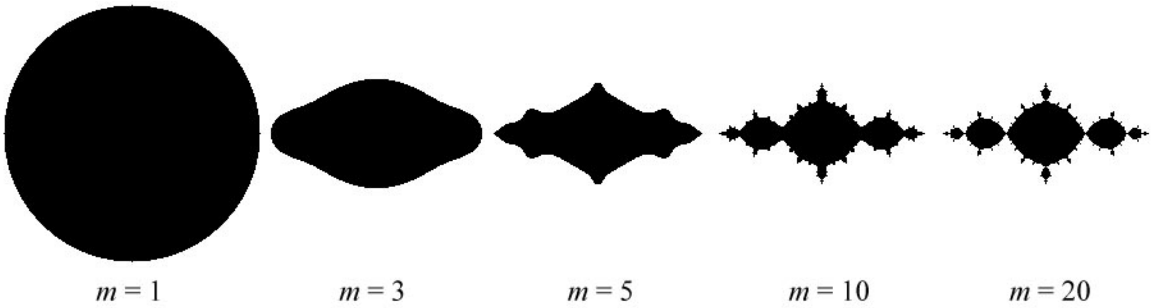


Fig.13 Approximation of the Julia set with $c = -1$ for different values of m

Similar activities can be done with other Julia sets, as well as with the Mandelbrot set.

Tasks:

[13] Try the same approximation for the filled Julia sets with $c = -1.25$ and then with $c = -1.4$. What can you observe?

[14] Try the approximation of the Mandelbrot set. How does the figure for $m = 1$ look? Why?

6.3 Life at the edge of chaos

Biology is starting to use chaos theory as well. And like in the apple man, the interesting areas seem to be the border regions. Some scientists believe that life develops on the *edge of chaos*: Too much chaos, and life cannot develop (if conditions drastically change all the time, most life forms cannot follow fast enough), too much stability and rigidity, and life cannot adjust enough to its environment and will become extinct.

6.4 Fractal art does not look artificial

As we have seen in some of the figures above, fractals are not "only" mathematical objects that one can study, but they can also be aesthetically very beautiful. So it is no wonder that several artists and designers use graphic representations of fractals in their work. And a lot of this work looks quite "natural", which, given the fractal structure of a lot of natural objects, does not come as a surprise. If you want to know more about the use of fractals in art and design, you might want to read [3]. A lot of examples of fractal art can be found in [4].

References

[1] http://www.dm.unipi.it/~olymp/comenius/ (August 9, 2011)

[2] http://upload.wikimedia.org/wikipedia/commons/a/a9/Julia-Teppich.png (October 14, 2011)

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