Using sliders to investigate functions, tangents and integrals

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1 Introduction

In many textbooks on differentiation there are exercises of the type where a student is asked to find the equation of the tangent line to the graph of a given function at a given value, e.g. at $x = 1$. Usually these problems are solved by differentiating the function, finding the value of the derivative at the given x – value e.g. $f\,'(1)$, to get the slope of the tangent line, and then finding the equation of a line through the given point e.g. $(1, f(1))$.

Software like GeoGebra or a graphics calculator is very useful in such situations to check if the calculated answer makes sense and also to connect the algebraic calculations just completed to the actual geometric object that has been found. It is possible to solve the problem in GeoGebra in two different ways, by computing the derivative and its value or by using the tangent tool. The derivative can be computed by writing $Derivative[f]$ or $f\,'(x)$ in the input field and its value at a given x – value by writing e.g. $f\,'(1)$ in the input field. It is also possible to use the *tangent tool*  to get the tangent at a certain point, directly. If that is done it is necessary to first define the point on the graph of the function by writing e.g. $(1, f(1))$ in the input field or by using the point tool  and clicking on the graph of the function.

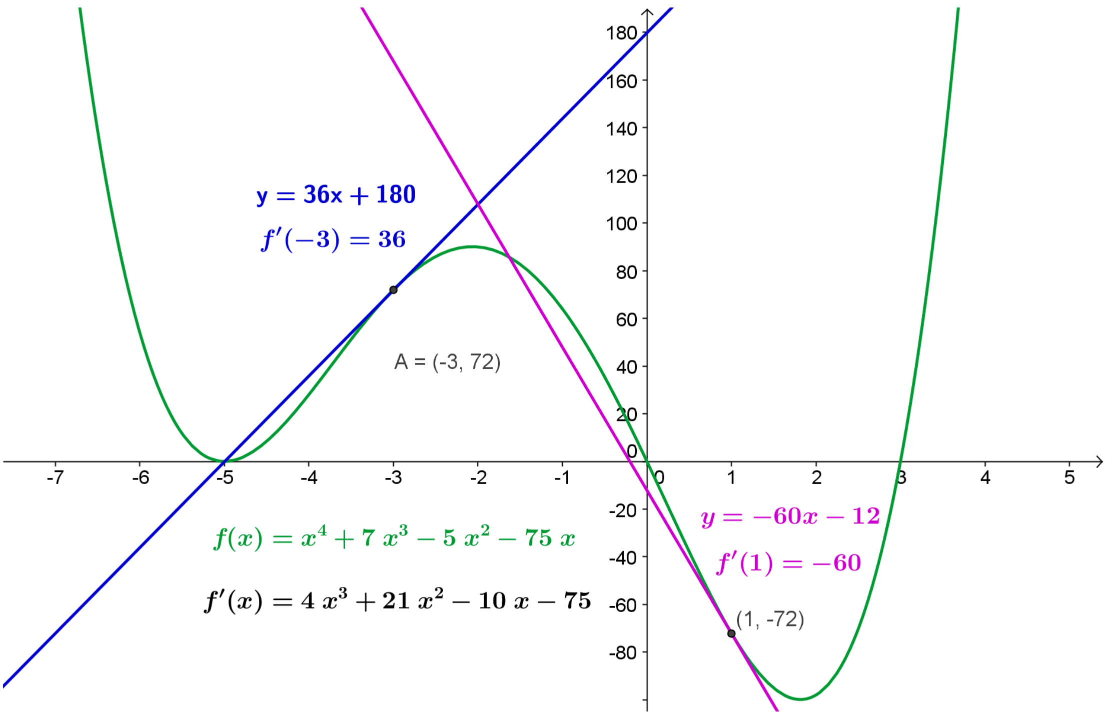


Fig. 1 Graph of a function with two of its tangent lines

GeoGebra can give extremal points of polynomials and can compute numerically extremal points for other functions if an interval is given (the command is *Extremum*) and there is also a command that gives points of inflection for polynomials (*InflectionPoint*).

A different kind of problem is of the type:

> *The slope of the tangent to* $f(x) = ax^3 - 3x^2 + x + 3$ *at the point* $(-1, f(-1))$ *is* 1.
>
> *Find the value of a.*

These problems are usually solved algebraically only, i.e. the student differentiates the function, substitutes the value -1 for *x* and solves $f'(-1) = 1$ to get the value of *a*:

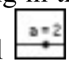
$f'(x) = 3ax^2 - 6x + 1$ which gives $f'(-1) = 3a + 6 + 1 = 1$ so $3a = -6 \Leftrightarrow a = -2$.

Once the problem has been translated to an algebra problem it is often not considered necessary graph the function or check the answer.

There are also in most calculus books more complicated problems of the same type involving the location of relative minimum and maximum, points of inflection, integrals etc. Most often the assumption is that the calculations are done by hand and the problem is converted into an algebra problem, i.e. into solving a system of equations. The focus is then more on algebra than on the actual shape of the function in question.

Below we shall consider ways to use GeoGebra to solve such problems in a more graphical way. Many of the tasks we consider come from material used in a gymnasium in Iceland [3].

2 Using sliders

Using GeoGebra it is very easy to investigate the effect of changing the value of one (or more) parameters occurring in the definition of a function such as the one above. To define such a parameter select the slider tool 

and click on the *Graphics view*. When that is done a small window opens:

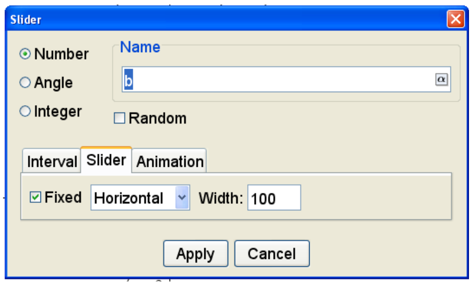


Fig. 2 This window is used for setting the interval of the parameter etc.

To solve the example given above we create a *slider a* and then define the function above by writing $f(x) = ax^3 - 3x^2 + x + 3$ in the input field. For GeoGebra to understand ax^3 we need to write either $a * x^3$ or $a\ x^3$.

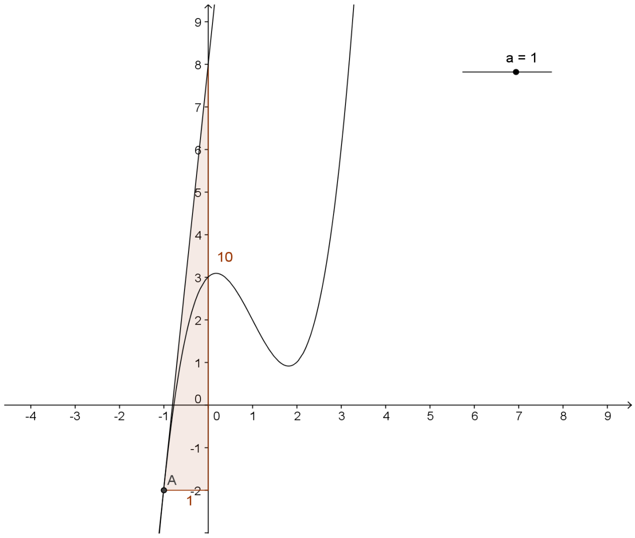


Fig. 3 The function *f(x)* with the original value of $a = 1$

In the figure above we have first defined the slider *a*, then the function $f(x)$. We then define the point $(-1, f(-1))$ in the input field (it gets the name *A)* and use the tangent tool  to get the tangent to $f(x)$ at that point. The *slope tool*  is used here to mark the slope of the tangent line. To find the desired value of a we simply move the slider until the value of the slope is 1.

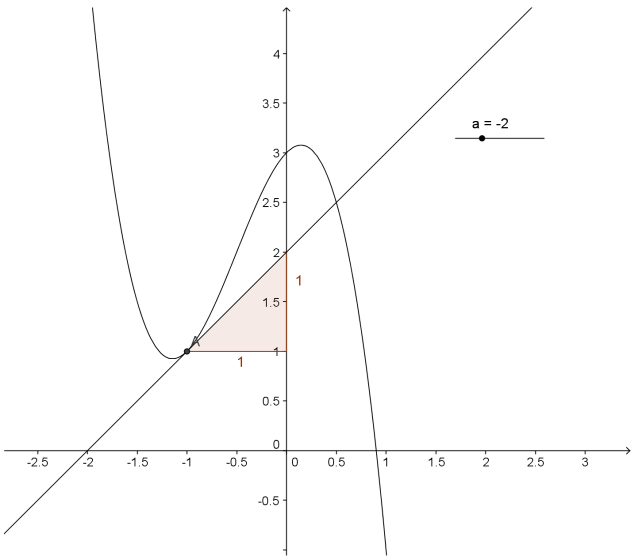


Fig. 4 The value $a = -2$ gives the desired slope

Task: The graph of the function $f(x) = x^2 + bx + c$ has a minimum at the point $(-3, -10)$ and goes through the point $(0, -1)$. Find the values of b and c by using sliders in GeoGebra (you will need to change the interval that b is defined on, this is done by right clicking on the slider and selecting *Object Properties)*.

Task: We have given the function $f(x) = x^2 - bx + c$. The tangent to $f(x)$ at the point $(-2, 0)$ is perpendicular to the line $y = x - 3$. Find the values of b and c using sliders in GeoGebra. Note that you have to think about the order in which you change the value of the sliders.

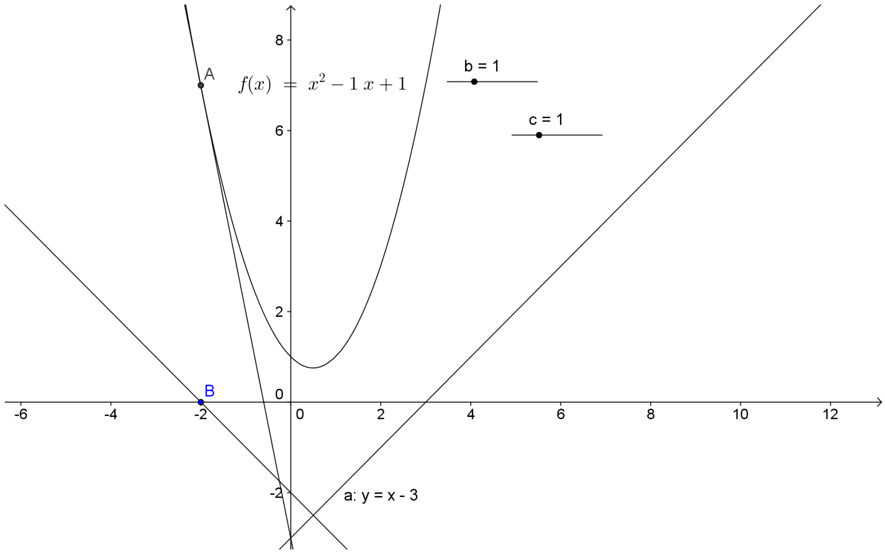


Fig. 5 Graph of $f\ (x) = x^2 - x + 1$ with a tangent line through the point $(-2, f(-2)) = (-2, 7)$. The line $y = x - 3$ together with a perpendicular line that goes through the point $(-2, 0)$.

3 Third degree polynomial

The problem below is from a handout for students at the gymnasium level [3]. It is assumed that the students will solve it using algebraic techniques after differentiation.
The objective is to find a third degree polynomial $f(x) = x^3 + bx^2 + cx + d$ that satisfies the following conditions:

The function has a minimum at $x = 1$ *and its graph has an inflection point at* $(-1\,, 12)$.

Below we take you through a number of steps to solve this problem using sliders in GeoGebra.

1. Define three sliders b, c and d. Define the third degree polynomial $f(x) = x^3 + bx^2 + cx + d$ in the input field.
   We can make this problem a little easier by making sure that the sliders only have integer values. To do this, right click on the slider and choose *Object Properties* and under the tab *Slider* set the *Increment* equal to 1.

2. Experiment with the sliders and try to find values such that the function satisfies the given conditions. In particular, what is the effect of changing the value of d?

3. Write $f\,'(x)$ in the input field, right click on the graph shown and choose *Properties* and change the color of the graph of $f\,'(x)$ so that it is easier to distinguish it from the graph of $f(x)$. Write $f\,''(x)$ in the input field and choose a third color for the appearing graph.

4. Change the value of c and observe its effect on the graph of $f\,'(x)$.

5. Change the value of b and observe its effect on the graph of $f\,''(x)$.

6. We have given that f has a point of inflection at $x = -1$. What does this tell us about the graph of $f\,''(x)$? Can you find a value of the slider b to satisfy this condition?

7. We have given that $f(x)$ has a minimum at $x = 1$. What does this tell us about the graph of $f'(x)$? Can you find a value of the slider c so that this is satisfied? Note that you might need to change the settings of the slider c, this is done by right clicking on c and choosing *properties*.

Now the graph of *f* should have the desired shape but we still need to find a value of *d* that gives the correct location of the graph.

8. Define the point $(-1, 12)$ in the input field. Change the value of d until the graph of $f(x)$ passes through the point. If this is difficult to see graphically you could write $f(-1)$ in the input field to get an exact value (in the algebra window) and then move d until that value is 12.

9. Now we should have found the correct values of *b*, *c* and *d* and the equation of the function $f(x)$ should be in the algebra window.

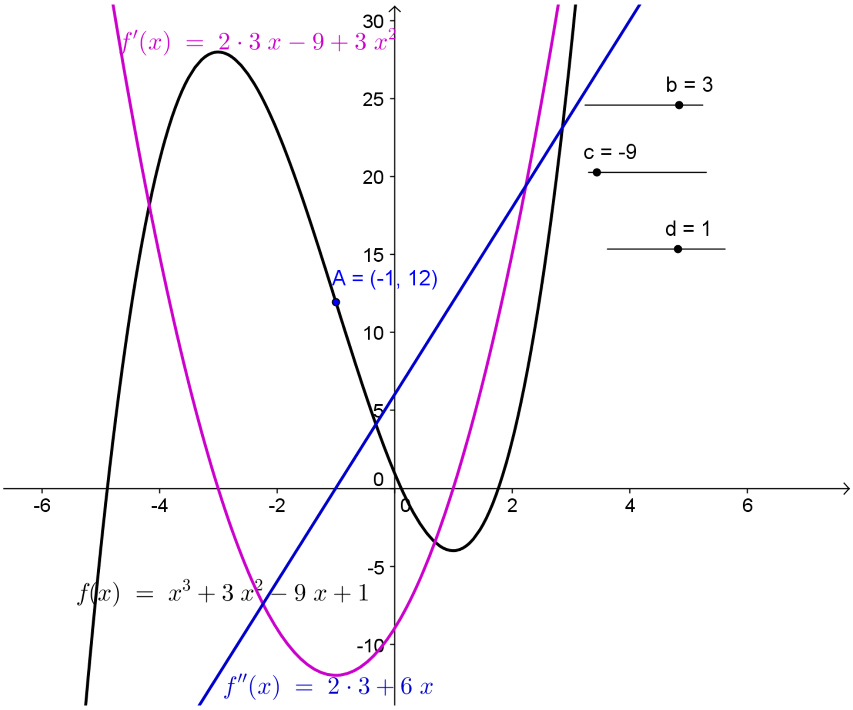


Fig. 6 The solution to the problem above

Task: If we are now asked to change the function such that the point of inflection is at $(-1, 10)$ which slider do we need to work on to accomplish that?

Task: What if the new information is that the point of inflection is at $(-2, 10)$?

Task: Can you change the values of the sliders such that *f(x)* does not have any extremal points? How?

Task: Can you change the values of the sliders such that *f(x)* does not have any inflection point? How?

A traditional way of solving the original problem would be to differentiate $f(x)$ twice and then use the information on the inflection point to get the equation $-6 + 2b = 0$, the information on the minimum to get $3 + 2b + c = 0$ and finally we get $-1 + b - c + d = 12$ since $(-1, 12)$ is on the graph.

Task: define one more slider a and redefine the polynomial as $f(x) = ax^3 + bx^2 + cx + d$. Set up the system of equations (with pencil and paper) we get if this function is to satisfy the conditions above. You should get 3 equations in 4 unknowns so there is a family of functions that satisfies the conditions. Can you redefine the parameters b, c and d in terms of a to get all third degree functions that satisfy the conditions?

Task: make up a similar problem for a fourth degree polynomial.

Task: make up similar problems using other types of functions e.g. exponential and logarithmic functions, trigonometric functions etc.

4 Problems with this solution method

Not every problem of this kind is this easy to solve. If we try to solve the similar problem:

Find the value of a such that $f(x) = ax^3 - 2x^2 - x + 5$ satisfies $f'(3) = 3$,

in the same way we do not get the solution as easily as in the previous examples.

Once we start doing what we did before we notice that it is not so easy to get a slope with exactly the value 3 i.e. the slope jumps between big and small values. It is therefore easier to click on the point on the slider and use the arrow keys on your keyboard to change the value of the slider. Doing this you see that the slope jumps between values a little bigger and a little smaller than 3. It is therefore wise to tune the slider more precisely and to do that we right click on a and choose *Object properties* and set the increment to 0.0001. Now we of course need to make sure that GeoGebra is actually using enough decimals; this is done under *Options* and then *Rounding.* With this increment we get that $a = 0.5926$ gives the slope 3.0002 and $a = 0.5925$ gives the slope 2.9975. The correct value of a is therefore somewhere between those two.

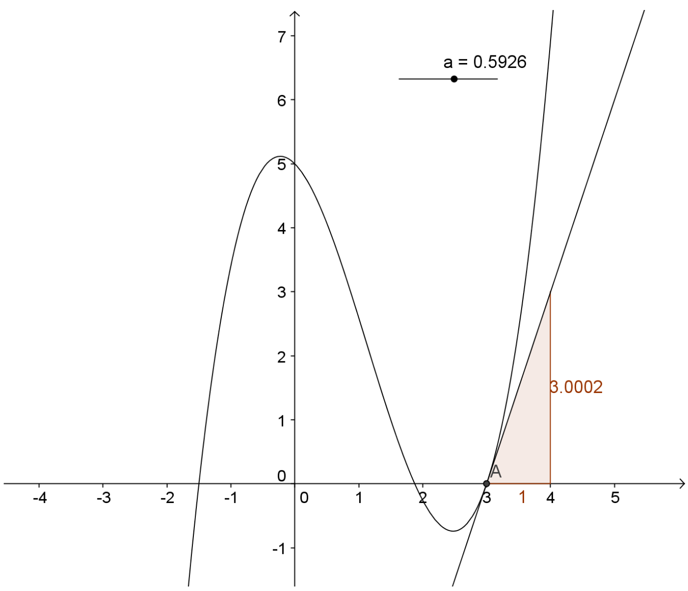


Fig. 7 Changing the increment of the slider

Task: try to explain what is happening. Hint: solve the problem algebraically.

Task: make the increment even smaller and try to solve the problem.

If we have enough patience to work on this using more decimals and a smaller increment we should eventually get that $a = 0.592592$ seems to get us very close to the desired values of the slope. The number appears close to a periodic decimal which we can convert to the fraction $a = 592/999 = 16/27$ which is exactly the solution we get from solving the problem algebraically.

5 Newton's law of cooling

Newton's law of cooling states that the rate of change of temperature of an object is proportional to the difference in temperature between the object and the surrounding medium. This leads to the differential equation $\frac{dT}{dt} = -k(T - T_0)$ where T is the temperature of the object, T_0 is the temperature of the surrounding medium and k is a constant. This equation has the solution

$$T = ce^{-kt} + T_0, \text{ where } c \text{ is a constant.}$$

So if a cup of coffee at the temperature 95℃ is placed in a room at 20℃ ($= T_0$) and 5 minutes later the coffee has the temperature 85℃ then we can use that information to find the values of c and k and thus get a model that will help us estimate when the coffee will be drinkable (say at 75℃).

This kind of a problem is usually solved algebraically, that is, we put the given information into the formula to get equations to solve:

when t = 0 we get $95 = ce^0 + 20$ so c = 75 and the formula is $T = 75e^{-kt} + 20$.

when t = 5 we get $85 = 75e^{-k \cdot 5} + 20$ which gives us $k = \frac{-1}{5} \ln\left(\frac{65}{75}\right) \approx 0.0286$

We therefore have $T = 75e^{-0.0286t} + 20$ and we can use this to find out when the coffee will have the temperature 75℃ (in approximately 11 minutes).

We can easily make a picture of this in GeoGebra:

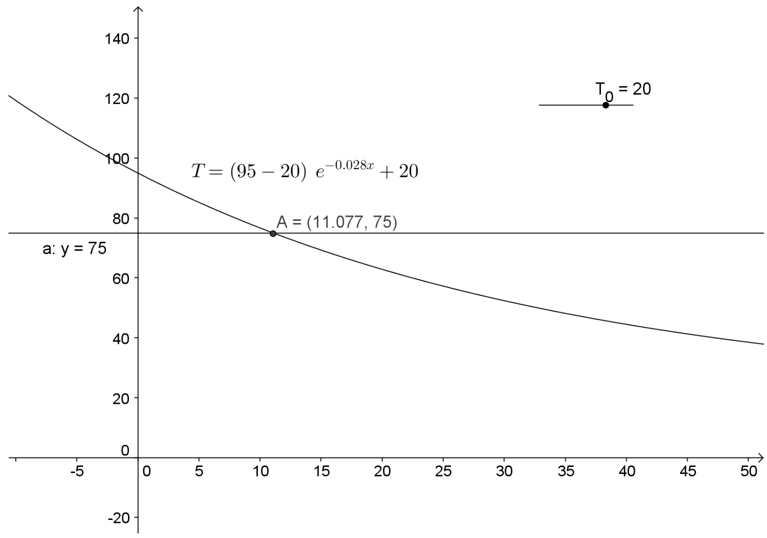


Fig. 8 Cooling of coffee

Instead of the calculation of k above we could have created a slider for k and then changed its value until the curve would pass through $(5, 85)$.

We can make this problem more interesting by varying the input information using sliders. In the figure above T_0 is a slider (although we didn't use it) but it does not really make sense to change its value only, because the information on the temperature of the coffee after 5 minutes should then also change as well as the value of k (if T_0 is less than 20℃ the temperature of the coffee would be less than 85℃ in *5* minutes).

We need three more sliders, T_c for the temperature of the object at the beginning, T_m for the information on temperature after m minutes and finally a slider m for minutes. A formula for k in terms of the given parameters is easy to write down and put in the input field of GeoGebra:

$$k = \frac{-1}{m} ln\left(\frac{T_m - T_0}{T_c - T_0}\right)$$

Task: Put the information above into GeoGebra. Assume now that you put your coffee cup on the windowsill where the temperature is 15℃ and after only 3 minutes the temperature is down to 85℃. When will the coffee be ready to drink?

Task: Reuse your model above for cooling of a Coca-Cola can that starts at room temperature. The temperature of a refrigerator is $2 - 3$ °C and the ideal temperature of Coca-Cola is $3 - 6$ °C (this is of course a matter of taste). Make some assumption for the values of m and T_m (you might want to experiment with this and compare with reality) and find out for how long you need to leave your Coca-Cola in the refrigerator. Now replace the refrigerator with the freezer (-18 °C) and solve the problem for that situation. Put both graphs in the same Graphics view to compare.

5 Sliders and Integration

Simple problems of integration can be solved directly in GeoGebra i.e. we can define a function and use the command *Integral[f(x)]* to get a primitive function and the command *Integral[f(x), a, b]* to get the definite integral from a to b. If we have two functions $f(x)$ and $g(x)$ then *IntegralBetween[f(x), g(x), a, b]* gives the integral of $f(x) - g(x)$ from a to b.

Typically sliders are used in this context to demonstrate how the values of the upper-sum and lower-sum approach the value of the integral when the number of intervals increases.

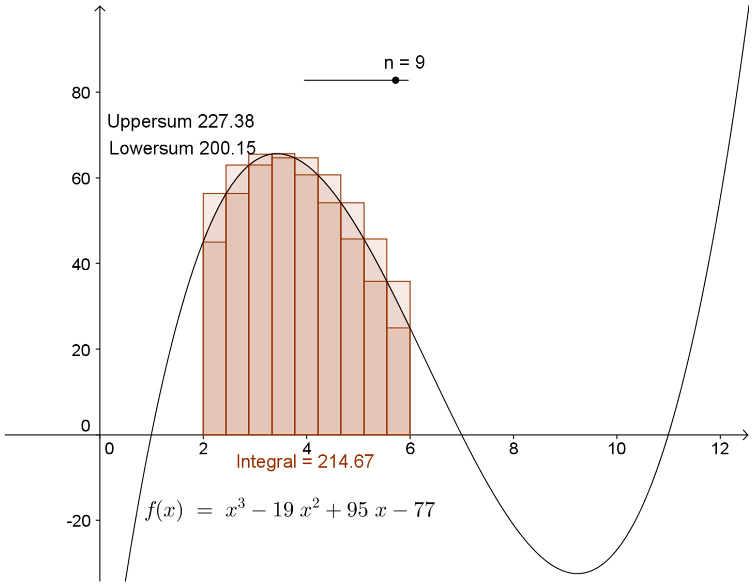


Fig. 9 Upper- and lower-sums and integral of a function

We can also use sliders to define dynamic intervals of integration.

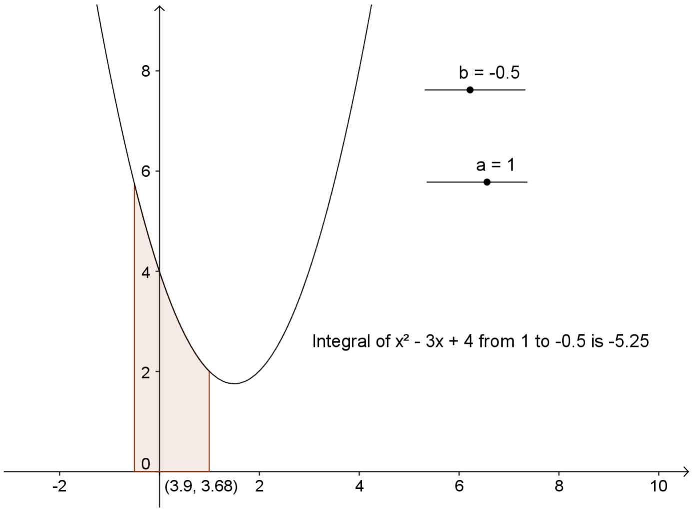


Fig. 10 Integrating

This can be useful to study certain problems of integration, say we have given a function $f(x)$ and we want to answer the question over which interval the integral of $f(x)$ equals a certain number. A typical way of solving such a problem would be to integrate $f(x)$, substitute boundaries a and b and then solve.

Using sliders a and b we can make GeoGebra compute the integral and then change the values of a and b until we get the desired value.

Below is an exercise from a Calculus book by Greenwell, Ritchey and Lial [2]:

Pollution begins to enter a lake at time $t = 0$ at a rate (in gallons per hour) given by the formula

$$f(t) = 10(1 - e^{-0.5t})$$

where t is the time in hours. At the same time, a pollution filter begins to remove the pollution at a rate

$$g(t) = 0.4t$$

as long as pollution remains in the lake.

The tasks in the problem are to a) determine the amount of pollution after 12 hours b) use a graphing calculator to find the time when the rate at which the pollution enters the lake equals the rate the pollution is removed c) find the amount of pollution in the lake at the time found in part b. d) Find the time when all the pollution has been removed from the lake.

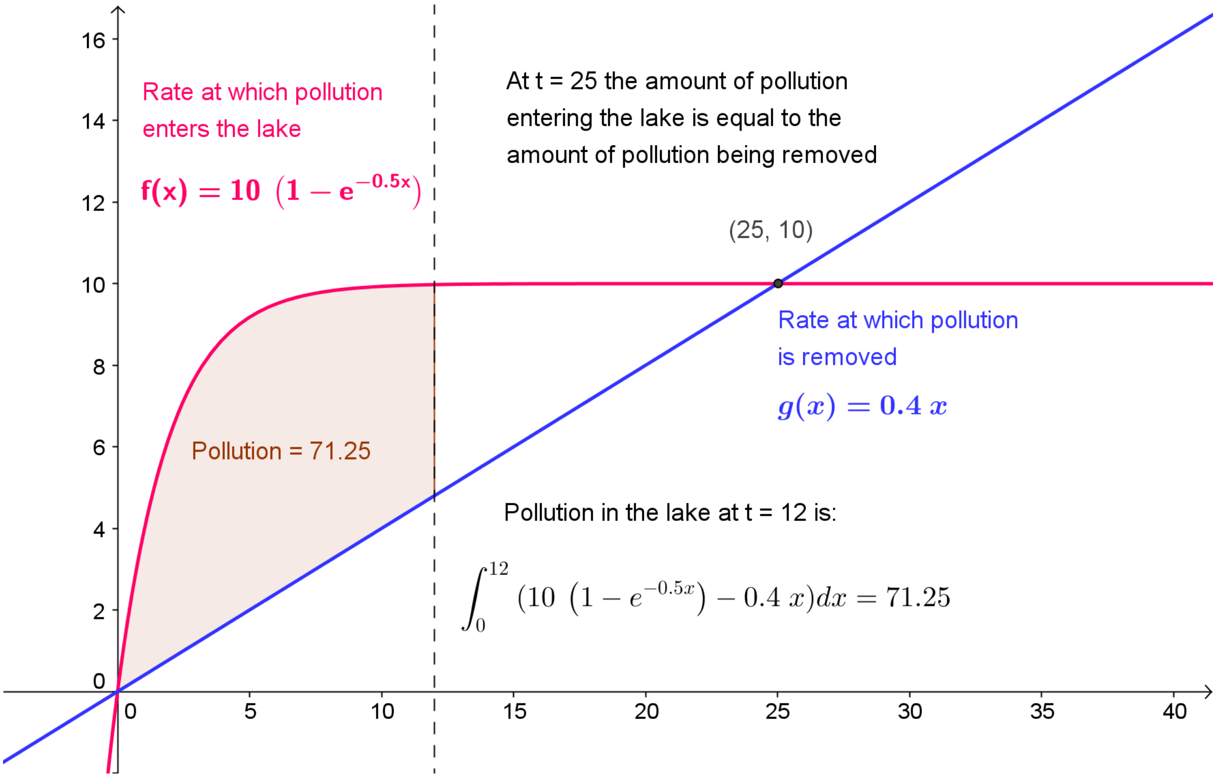


Fig. 11 Part a and b of the pollution problem

Part a is solved by integrating *f(x) – g(x)* from 0 to 12 and part b by finding the point of intersection for the two graphs. Part c is solved by integrating from 0 to 25, which gives 105 gallons of pollution. After t = 25 hours the pollution is being cleaned up at a faster rate than it enters so the amount of pollution decreases, for instance at t = 30 hours there are 100 gallons of pollution. To find out when all the pollution has been cleaned up we need to find the upper limit L of the integral such that

$$\int_0^L \big(f(t) - g(t)\big)dt = \int_0^L (10(1 - e^{-0.5t}) - 0.4t)dt = 0$$

This can of course be solved by integrating the function by hand and solving for L but it is very convenient to use a slider here, i.e. we define a slider L, calculate the integral from 0 to L and move the slider until we get an answer that is sufficiently close to 0.

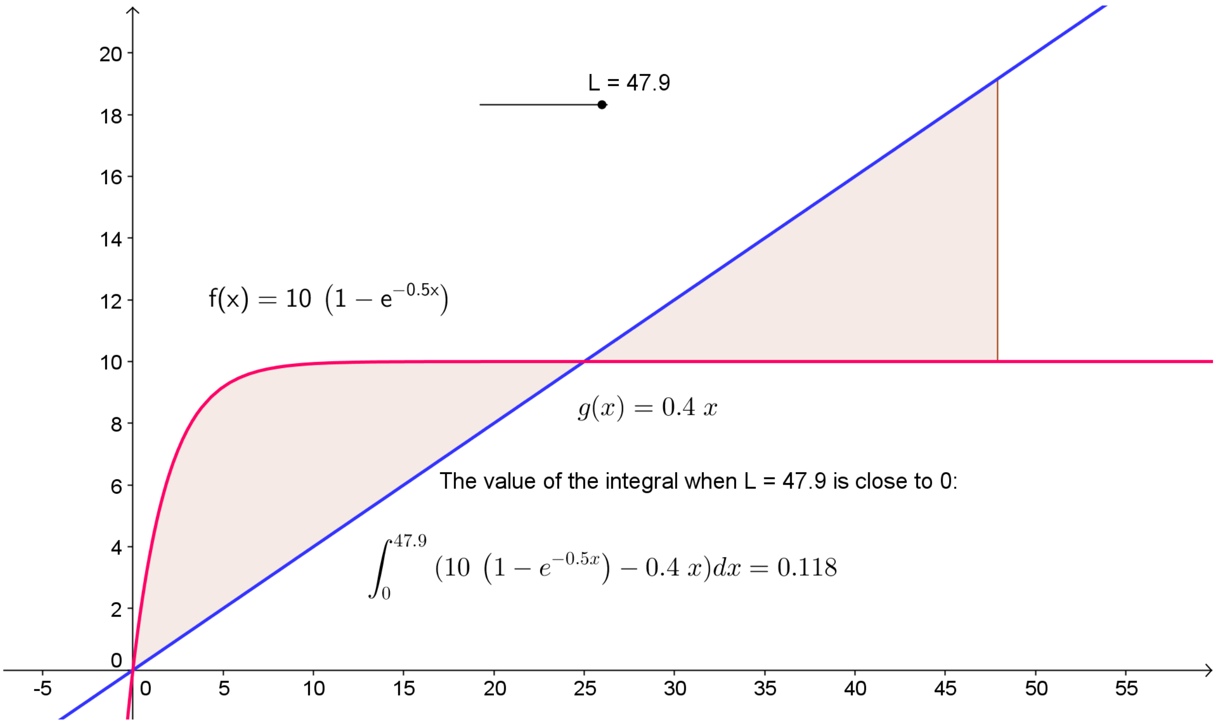


Fig. 12 After 48 hours the cleanup process has caught up with the pollution and cleaned up the accumulated pollution.

Task: solve the problem above if the pollution rate is doubled.

Task: assuming that the cleanup rate is linear, what does it have to be so that we can get the situation under control in less than 24 hours? (hint: define a slider for the coefficient of g(x)).

References

[1] GeoGebra, downloadable from http://www.geogebra.org.

[2] Greenwell, R. N. , Ritchey, N.P. and Lial, M.L. (2003), *Calculus for the Life Sciences,* USA: Pearson Education.

[3] Stærðfræði 403, Menntaskólinn við Hamrahlíð (2010).