Perimeter of harmonic triangles in ellipse

Vladimir Georgiev, Veneta Nedyalkova

1 Short introduction

We continue our study of some properties of periodic triangles on billiard tables with the following argument presented on page 170 in [1]:

"Next we ask for the triangle of maximum length inscribed in a C(C is a curve on a plane). Evidently at least one such triangle will exist, and can have no degenerate side of length 0. At each of its vertices the tangent will of course make equal angles with the two sides passing through the vertex. Hence a 'harmonic triangle' is obtained which will correspond to two distinct motions, one for each of the two possible senses of description. Moreover, if we seek to vary this triangle continuously, not changing the order of its vertices and diminishing the perimeter as little as possible, so as finally to advance the vertices cyclically, we discover a second harmonic triangle, also corresponding to two periodic motions. "

2 Explicit examples of concrete harmonic triangles

As a simplest case we choose the ellipse

$$e: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

and take the point $A_0 \in e$ on the $y-$ axis, i.e. $A_0(0, b)$ (see Figure 1).

Following the construction suggested by Birkhoff ([1]) one can find the triangle $\triangle\, A_0A_1A_2$ with maximal perimeter inscribed in e.

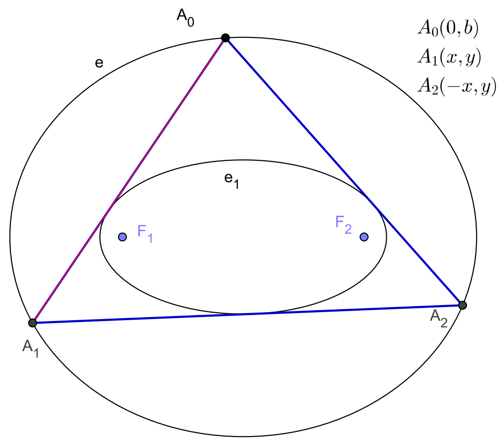

Figure 1: Harmonic triangle with $A_0(0, b)$.

In [5] we found explicit expressions for the coordinates of the points $A_1(x, y), A_2(-x, y)$.

Recalling the assertion of Theorem 1 in [5], one can see that if $\triangle\, A_0A_1A_2$ is periodic, then its caustic is a confocal ellipse, say

$$e_1: \quad \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1. \tag{2}$$

The equation (6) shows now that the condition that e and e_1 are confocal, i.e. have the same foci F_1 and F_2 can be expressed by

$$a^2 - b^2 = a_1^2 - b_1^2. \tag{3}$$

We shall suppose the e_1 is inside e so we have

$$a > b > 0, a_1 > b_1 > 0, a > a_1, b > b_1.$$

Recall some of the results in [5].

Lemma 1. *(see [5]) Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ one can express the necessary and sufficient condition such that the line $y = kx + b$ through the point $A_0(0, b)$ is tangent to e_1 as follows*

$$k = \pm\frac{b_2 - b_1^2}{a_1^2}.$$

Lemma 2. *(see [5]) Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(0, b)$ denote by t_1, t_2 the tangent lines from A_0 to e_1 and by A_1, A_2 the points of the intersection of these tangent lines with the ellipse $e : x^2/a^2 + y^2/b^2 = 1$, such that $A_1(x, y), x < 0$, $A_2(-x, y)$. Then we have*

$$x = \frac{2a^2 a_1 b\sqrt{b^2 - b_1^2}}{a_1^2 b^2 + a^2(b^2 - b_1^2)} = -\frac{2bka^2}{b^2 + a^2k^2},$$

$$y = -M, \quad M = \frac{a^2 b(b^2 - b_1^2)}{a_1^2 b^2 + a^2(b^2 - b_1^2)} = \frac{b(b^2 - a^2k^2}{b^2 + a^2k^2}.$$

From the relation $M = b_1$ one can find a_1, b_1 taking into account the fact that e and e_1 are confocal.

Lemma 3. *Given an ellipse $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(0, b)$ let $A_1(x_1, y_1), x_1 < 0$, $A_2(x_2, y_2), x_2 > 0$ are the points determined in Lemma 2. Then A_1A_2 is tangent to e_1 if and only if*

$$a_1 = \frac{a(\sqrt{a^4 - a^2b^2 + b^4} - b^2)}{a^2 - b^2},$$

$$b_1 = \frac{b(a^2 - \sqrt{a^4 - a^2b^2 + b^4})}{a^2 - b^2}.$$

Proof. The relation $M = b_1$ is equivalent to

$$\frac{b(b^2 - a^2k^2}{b^2 + a^2k^2} = b^1.$$

This can be rewritten in the form

$$b^2(b - b_1) - a^2k^2(b + b_1)$$

so using Lemma 1 we get

$$b^2(b - b_1) - \frac{a^2(b - b_1)(b + b_1)^2}{a_1^2} = 0.$$

From $b \neq b_1$ we simplify

$$b^2 - \frac{a^2(b+b_1)^2}{a_1^2} = 0$$

or

$$a_1^2 b^2 = a^2(b+b_1)^2$$

and this means that

$$a_1 b = a(b+b_1).$$

This relation and the fact that e, e_1 are confocal leads to the system

$$\begin{cases} a_1 b = a(b+b_1), \\ a_1^2 - b_1^2 = a^2 - b^2. \end{cases} \tag{4}$$

This system has unique solution $a_1 > 0, b_1 > 0$ determined by

$$a_1 = \frac{a(\sqrt{a^4 - a^2b^2 + b^4} - b^2)}{a^2 - b^2},$$

$$b_1 = \frac{b(a^2 - \sqrt{a^4 - a^2b^2 + b^4})}{a^2 - b^2}.$$

This completes the proof of the Lemma.

□

One can introduce the quantity

$$s = \frac{b}{a} \in (0,1). \tag{5}$$

Then we have the following representation formulas for a_1, b_1

$$a_1 = a\frac{(\sqrt{1 - s^2 + s^4} - s^2)}{1 - s^2},$$

$$b_1 = b\frac{(1 - \sqrt{1 - s^2 + s^4})}{1 - s^2}.$$

One can try to change the point A_0 choosing $A_0(a,0)$ (see Figure 2).

Exercise 1. *Given an ellipse* $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ *one can express the necessary and sufficient condition such that the line* $y = kx - ka$ *through the point* $A_0(a,0)$ *is tangent to* e_1 *as follows*

$$k = \pm\frac{b_2 - b_1^2}{a_1^2}.$$

Exercise 2. *Given an ellipse* $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ *and the point* $A_0(a,0)$ *denote by* t_1, t_2 *the tangent lines from* A_0 *to* e_1 *and by* A_1, A_2 *the points of the intersection of these tangent lines with the ellipse* $e : x^2/a^2 + y^2/b^2 = 1$, *such that* $A_1(x,y), x < 0$, $A_2(x,-y)$. *Then we have*

$$x = -\frac{2a(k^2 - b^2)}{b^2 + a^2k^2},$$

$$y = \frac{2ab_2k}{b^2 + a^2k^2}.$$

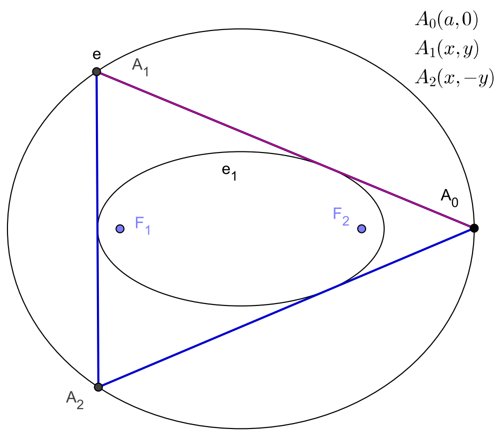

Figure 2: Harmonic triangle with $A_0(a,0)$.

Following the proof of Lemma 3 one can solve the following exercise.

Exercise 3. *Given an ellipse* $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ *and the point* $A_0(0,b)$ *let* $A_1(x_1,y_1), x_1 < 0$, $A_2(x_2,y_2), x_2 > 0$ *are the points determined in Lemma 2. Then* A_1A_2 *is tangent to* e_1 *if and only if*

$$a_1 = \frac{a(\sqrt{a^4 - a^2b^2 + b^4} - b^2)}{a^2 - b^2},$$

$$b_1 = \frac{b(a^2 - \sqrt{a^4 - a^2b^2 + b^4})}{a^2 - b^2}.$$

3 Perimeter of concrete harmonic triangles

In the simplest case of the ellipse

$$e : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{6}$$

and point $A_0 \in e$ on the $y-$ axis, i.e. $A_0(0,b)$ (see Figure 1) we have explicit formulas for $A_1(x,y), A_2(-x,y)$

$$x = -\frac{2bka^2}{b^2 + a^2k^2},$$

$$y = -M, \quad M = \frac{b(b^2 - a^2k^2}{b^2 + a^2k^2},$$

obtained in Lemma 2. The perimeter of the triangle $\triangle\, A_0A_1A_2$ is

$$P_1 = 2\sqrt{x^2 + (y-b)^2} + 2|x| =$$

$$= \frac{4a^2b|k|\sqrt{k^2+1}}{b^2 + a^2k^2} + \frac{4b|k|a^2}{b^2 + a^2k^2}.$$

Lemma 1 can be used to find the expression for P_1 and to prove the following.

Lemma 4. *Given an ellipse* $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ *and the point* $A_0(0, b)$ *let* $A_1(x_1, y_1), x_1 < 0$, $A_2(x_2, y_2), x_2 > 0$ *are the points determined in Lemma 2. Then the perimeter* P_1 *of the triangle* $\triangle\ A_0A_1A_2$ *is given by*

$$P_1 = \frac{4a^2b(a + a_1)\sqrt{a^2 - a_1^2}}{b^2a_1^2 + a^2(a^2 - a_1^2)}.$$

Moving the point A_0 so that $A_0(a, 0)$ we can prove the next

Lemma 5. *Given an ellipse* $e_1 : x^2/a_1^2 + y^2/b_1^2 = 1$ *and the point* $A_0(a, 0)$ *let* $A_1(x_1, y_1), x_1 < 0$, $A_2(x_2, y_2), x_2 > 0$ *are the points determined in Exercise 2. Then the perimeter* P_2 *of the triangle* $\triangle\ A_0A_1A_2$ *is given by*

$$P_2 = \frac{4ab^2(b + b_1)\sqrt{a^2 - a_1^2}}{b_1^2a^2 + b^2(a^2 - a_1^2)}.$$

Exercise 4. *Show that* $S_1 = S_2$.

Hint. Use (5).

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