

Dyna MAT

Perimeter of harmonic triangles in ellipse Modelling optical lenses with Dynamic Geometry Software

Vladimir Georgiev, Veneta Nedyalkova Andreas Ulovec

1 Short introduction

We continue our study of some properties of periodic triangles on billiard tables with the following argument presented on page 170 in [1]: lenses were commune our sourcy or some properties or periodic triangles on bimard tables with the

"Next we ask for the triangle of maximum length inscribed in a $C(C$ is a curve on a plane). Evidently at least one such triangle will exist, and can have no degenerate side of length 0. At each of its vertices the tangent will of course make equal angles with the two sides passing At each of its vertices the tangent will of course make equal angles with the two sides passing through the vertex. Hence a 'harmonic triangle' is obtained which will correspond to two distinct motions, one for each of the two possible senses of description. Moreover, if we seek to vary this triangle continuously, not changing the order of its vertices and diminishing the perimeter as little as possible, so as finally to advance the vertices cyclically, we discover a second harmonic triangle, also corresponding to two periodic motions. " the call of its vertices the taigent will of course make equal angles with the two sides pass

2 Explicit examples of concrete harmonic triangles ϵ - explicit examples of concrete narihonic triangles

As a simplest case we choose the ellipse and the glass and the glass and continues the same happens when the

$$
e: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
$$
 (1)

and take the point $A_0 \in e$ on the $y-$ axis, i.e. $A_0(0, b)$ (see Figure 1).

Following the construction suggested by Birkhoff ([1]) one can find the triangle $\Delta A_0 A_1 A_2$ with maximal perimeter inscribed in e . become more complex, and from the equations alone it would be difficult to see what happens. With Following the construction suggested by Birkhoff ([1]) one can find the triangle Δ $A_0A_1A_2$ w

Figure 1: Harmonic triangle with $A_0(0, b)$.

In [5] we found explicit expressions for the coordinates of the points $A_1(x, y)$, $A_2(-x, y)$.

Recalling the assertion of Theorem 1 in [5], one can see that if $\Delta A_0A_1A_2$ is periodic, then its caustic is a confocal ellipse, say

$$
e_1: \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1. \tag{2}
$$

The equation (6) shows now that the condition that *e* and e_1 are confocal, i.e. have the same foci F_1 and F_2 can be expressed by **Modelling optical lenses with Dynamic Geometry Software**

$$
a^2 - b^2 = a_1^2 - b_1^2. \tag{3}
$$

We shall suppose the e_1 is inside e so we have $\frac{1}{2}$ is the experiment of the experiments are $\frac{1}{2}$ in the experiment complex, and you need a lot of $\frac{1}{2}$ in the experiment of $\frac{1}{2}$

$$
a > b > 0, a_1 > b_1 > 0, a > a_1, b > b_1.
$$

Recall some of the results in $[5]$.

Lemma 1. *(see [5])* Given an ellipse e_1 : $x^2/a_1^2 + y^2/b_1^2 = 1$ one can express the necessary and sufficient condition such that the line $y = kx + b$ through the point $A_0(0, b)$ is tangent to e_1 as *follows* \mathbf{r} thicker, you have to take out the new one. Students can then observe the new one. Students can then observe **Lemma 1.** (see [b]) Given an ellipse $e_1: x^2/a_1^2 + y^2/b_1^2 = 1$ one can express the necessary of Ω through a lens with the help of dynamic geometry software (Ω GS).

$$
k = \pm \frac{b_2 - b_1^2}{a_1^2}.
$$

Lemma 2. (see [5]) Given an ellipse e_1 : $x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(0,b)$ denote by t_1, t_2 the tangent lines from A_0 to e_1 and by A_1, A_2 the points of the intersection of these tangent lines with the ellipse $e: x^2/a^2 + y^2/b^2 = 1$, such that $A_1(x, y), x < 0, A_2(-x, y)$. Then we have **formation for the mathematics teachers.** The mathematics is a lot of it is a lot of it is a ratio of α ratio α ra

$$
x = \frac{2a^2a_1b\sqrt{b^2 - b_1^2}}{a_1^2b^2 + a^2(b^2 - b_1^2)} = -\frac{2bka^2}{b^2 + a^2k^2},
$$

$$
y = -M, \quad M = \frac{a^2b(b^2 - b_1^2)}{a_1^2b^2 + a^2(b^2 - b_1^2)} = \frac{b(b^2 - a^2k^2)}{b^2 + a^2k^2}.
$$

From the relation $M = b_1$ one can find a_1, b_1 taking into account the fact that *e* and e_1 are confocal.

Lemma 3. Given an ellipse $e_1: x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(0, b)$ let $A_1(x_1, y_1), x_1 < 0$, $A_2(x_2, y_2), x_2 > 0$ are the points determined in Lemma 2. Then A_1A_2 is tangent to e_1 if and *only if* $tanh$ if the ray of light and the *normal*) is equal to the angle of n

$$
a_1 = \frac{a(\sqrt{a^4 - a^2b^2 + b^4} - b^2)}{a^2 - b^2},
$$

$$
b_1 = \frac{b(a^2 - \sqrt{a^4 - a^2b^2 + b^4})}{a^2 - b^2}.
$$

Proof. The relation $M = b_1$ is equivalent to

$$
\frac{b(b^2 - a^2k^2)}{b^2 + a^2k^2} = b^1.
$$

This can be rewritten in the form

$$
b^2(b - b_1) - a^2k^2(b + b_1)
$$

so using Lemma 1 we get

$$
b^{2}(b - b_{1}) - \frac{a^{2}(b - b_{1})(b + b_{1})^{2}}{a_{1}^{2}} = 0.
$$

Dyna MAT

From $b \neq b_1$ we simplify **Modelling optical lenses with Dynamic Geometry Software** Andreas Ulovec

$$
b^{2} - \frac{a^{2}(b + b_{1})^{2}}{a_{1}^{2}} = 0
$$
or
$$
a_{1}^{2}b^{2} = a^{2}(b + b_{1})^{2}
$$

or 1 **1 International**

and this means that \mathcal{L} this maans that $\frac{1}{2}$ is different. It is different enough to show a ray of light in air – you need smoke, dust or any other way of light in air – you need smoke, dust or any other way of light in any other way of light in any other

 $a_1b = a(b + b_1).$ $u_1v=u(v+v_1).$

This relation and the fact that e, e_1 are confocal leads to the system $\frac{d}{dt}$ removing and putting and putting another piece in. To see what happens if $\frac{d}{dt}$

$$
\begin{cases}\n a_1 b = a(b + b_1), \\
a_1^2 - b_1^2 = a^2 - b^2.\n\end{cases}
$$
\n(4)

This system has unique solution $a_1 > 0, b_1 > 0$ determined by

$$
a_1 = \frac{a(\sqrt{a^4 - a^2b^2 + b^4} - b^2)}{a^2 - b^2},
$$

$$
b_1 = \frac{b(a^2 - \sqrt{a^4 - a^2b^2 + b^4})}{a^2 - b^2}.
$$

This completes the proof of the Lemma. which the light is reflected and reflection and reflection \mathbf{F}

One can introduce the quantity

$$
s = \frac{b}{a} \in (0, 1). \tag{5}
$$

Then we have the following representation formulas for a_1, b_1

$$
a_1 = a \frac{(\sqrt{1 - s^2 + s^4} - s^2)}{1 - s^2},
$$

$$
b_1 = b \frac{(1 - \sqrt{1 - s^2 + s^4})}{1 - s^2}.
$$

One can try to change the point A_0 choosing $A_0(a, 0)$ (see Figure 2).

Exercise 1. *Given an ellipse* e_1 : $x^2/a_1^2 + y^2/b_1^2 = 1$ *one can express the necessary and sufficient condition such that the line* $y = kx - ka$ *through the point* $A_0(a, 0)$ *is tangent to* e_1 *as follows*

$$
k = \pm \frac{b_2 - b_1^2}{a_1^2}.
$$

Exercise 2. Given an ellipse e_1 : $x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(a, 0)$ denote by t_1, t_2 the *tangent lines from* A_0 *to* e_1 *and by* A_1 , A_2 *the points of the intersection of these tangent lines with the ellipse* $e: x^2/a^2 + y^2/b^2 = 1$ *, such that* $A_1(x, y), x < 0, A_2(x, -y)$ *. Then we have*

$$
x = -\frac{2a(k^{2} - b^{2})}{b^{2} + a^{2}k^{2}},
$$

$$
y = \frac{2ab_{2}k}{b^{2} + a^{2}k^{2}}.
$$

 \Box

Figure 2: Harmonic triangle with $A_0(a, 0)$. f_{tot} mathematic strange with $f_{\text{U}}(\omega, \nu)$.

Following the proof of Lemma 3 one can solve the following exercise. light reaches the other surface of the lens – again mathematics is required to calculate the angle in

Exercise 3. Given an ellipse $e_1: x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(0, b)$ let $A_1(x_1, y_1), x_1 < 0$, $A_2(x_2, y_2), x_2 > 0$ are the points determined in Lemma 2. Then A_1A_2 is tangent to e_1 if and *only if* near the centre of the lenses and light being more of ϵ and light being more of ϵ $\frac{a\left(\sqrt{a^4-a^2b^2+b^4}-b^2\right)}{a}$

$$
a_1 = \frac{a(\sqrt{a^4 - a^2b^2 + b^4} - b^2)}{a^2 - b^2},
$$

$$
b_1 = \frac{b(a^2 - \sqrt{a^4 - a^2b^2 + b^4})}{a^2 - b^2}.
$$

3 Perimeter of concrete harmonic triangles 2.1 Reflection When a ray of light hits a plane glass surface, a part of it is reflected. The *law of reflection* says that

In the simplest case of the ellipse

$$
e: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{6}
$$

and point $A_0 \in e$ on the *y−* axis, i.e. $A_0(0, b)$ (see Figure 1) we have explicit formulas for *A*1(*x, y*)*, A*2(*−x, y*)

$$
x = -\frac{2bka^2}{b^2 + a^2k^2},
$$

$$
y = -M, \quad M = \frac{b(b^2 - a^2k^2)}{b^2 + a^2k^2},
$$

obtained in Lemma 2. The perimeter of the triangle $\Delta A_0A_1A_2$ is

$$
P_1 = 2\sqrt{x^2 + (y - b)^2} + 2|x| =
$$

=
$$
\frac{4a^2b|k|\sqrt{k^2 + 1}}{b^2 + a^2k^2} + \frac{4b|k|a^2}{b^2 + a^2k^2}.
$$

Lemma 1 can be used to find the expression for P_1 and to prove the following.

Dyna MAT

Lemma 4. Given an ellipse e_1 : $x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(0,b)$ let $A_1(x_1,y_1), x_1 < 0$, $A_2(x_2, y_2), x_2 > 0$ are the points determined in Lemma 2. Then the perimeter P_1 of the triangle $\Delta A_0 A_1 A_2$ *is given by* **Modelling optical lenses with Dynamic Geometry Software 1 Introduction**

$$
P_1 = \frac{4a^2b(a+a_1)\sqrt{a^2 - a_1^2}}{b^2a_1^2 + a^2(a^2 - a_1^2)}.
$$

Moving the point A_0 so that $A_0(a, 0)$ we can prove the next

Lemma 5. Given an ellipse $e_1: x^2/a_1^2 + y^2/b_1^2 = 1$ and the point $A_0(a, 0)$ let $A_1(x_1, y_1), x_1 < 0$, $A_2(x_2, y_2), x_2 > 0$ are the points determined in Exercise 2. Then the perimeter P_2 of the triangle $\Delta A_0A_1A_2$ *is given by*

$$
P_2 = \frac{4ab^2(b+b_1)\sqrt{a^2 - a_1^2}}{b_1^2a^2 + b^2(a^2 - a_1^2)}.
$$

Exercise 4. *Show that* $S_1 = S_2$.

Hint. Use (5) . for mathematics teachers. Well, now where is the mathematics? There is a lot of it in there! If a ray of

References surface of an optical lens, a part of an optical lens, a part of it gets reflected back in a certain angle, and angle, and an

- [1] G D. Birkhoff, *Dynamical systems*, AMS, Coll. Publ. Vol. 9, Revised edition (1966). another part penetrates the glass and continues there, in another angle. The same happens when the
- [2] A. Cayley, *Note on the porism of the in-and-circumscribed polygon*, Philosophical magazine **6** (1853), 99102. [2] A. Cayley, *ivote on the portsm of the in-ana-circumscribea polygon*, Philosophical magaz
- [3] A. Cayley, *Developments on the porism of the in-and-circumscribed polygon*, Philosophical magazine **7** (1854), 339345. \lbrack o Eugene, possible to simulate the portion of the \lbrack in and circumscribed polygon, Philosophi
- [4] V. Dragovic, M. Radnovic, *Poncelet Porisms and Beyond*, Birkhäuser, Springer-Basel, 2011.
- [5] V.Georgiev, I.Georgieva, V.Nedyalkova, *Dynamical billiards*, article in this book.
- [6] V.Georgiev, V.Nedyalkova, *Poncelet's porism and periodic triangles in ellipse*, article in this book. the book.
- [7] S.Tabachnikov, *Geometry and Billiards*, Students Mathematica Library, (2005)